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Exact Quantization of a Superparticle in $AdS_5 \times S^5$

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Abstract

As a step toward deeper understanding of the AdS/CFT correspondence, exact quantization of a Brink-Schwarz superparticle in the $AdS_5 \times S^5$ background with Ramond-Ramond (RR) flux is performed from the first principle in the phase space formulation. It includes the construction of the quantum Noether charges for the $psu(2, 2|4)$ superconformal symmetry and by solving the superconformal primary conditions we obtain the complete physical spectrum of the system with the explicit wave functions. The spectrum agrees precisely with the supergravity results, including all the Kaluza-Klein excitations. Our method and the result are expected to shed light on the eventual quantization of a superstring in this important background.

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1 Introduction

For more than a decade since its inception, the concept of AdS/CFT[1, 2, 3] has been an inexhaustible source of new developments in both string theory and quantum field theory. In recent years it has been applied to such broad areas as QCD phenomenology[4, 5, 6], condensed matter physics[7] and so on that if successful its magical power would be even more enhanced. It is “magical” since, despite the existence of a pile of impressive evidence, the understanding of the fundamental mechanism of this correspondence is still a difficult unsolved problem.

Evidently, the major reason for this difficulty lies in the strong/weak nature of the correspondence. In the prototypical example of the correspondence between the $N = 4$ super-Yang-Mills (SYM) theory in 4 dimensions and the type IIB superstring in $AdS_5 \times S^5$ with RR flux, which will be the exclusive focus of our attention in this article, it is expressed by the well-known relation $g_{YM}^2 N = 4\pi g_s N = R^4/\alpha'^2$, where R is the common radius of AdS_5 and S^5 . This succinctly expresses the fact that large 't Hooft coupling on the CFT side corresponds to the weak coupling on the string worldsheet and vice versa. To understand the physical meaning of this relation, one notes that it contains two equalities of different nature. The first equality signifies the familiar open-closed duality, which holds perturbatively. The second equality on the other hand refers only to the closed string side. It can be interpreted as expressing the fact that the metric and the RR 5-form condense in tandem to produce the $AdS_5 \times S^5$ with common radius R . In fact the action density of the metric is given by $\mathcal{R}/(g_s^2 l_s^8) \sim 1/(g_s^2 l_s^8 R^2)$, where \mathcal{R} is the scalar curvature and l_s is the string scale, while the action density of the N units of 5-form flux is $F_5^2 \sim (N/R^5)^2$, where R^5 is the volume of S^5 . Equating these two expressions one immediately obtains (the square of) the second equality. This suggests that to understand the AdS/CFT correspondence dynamically one would have to sum over the infinite number of open string loop diagrams attached to a stack of D-branes and interpret it from the closed string channel as condensation of the metric and the RR 5-form which warps the spacetime. In other words, it tantamounts to showing rigorously that the D-branes are what we believe they are. Attempts along this line have been made recently[8, 9][10, 11], but a precise tractable formulation appears to be hard at the moment.

In short of the fundamental dynamical understanding, the next best thing is to demonstrate that the symmetry structure, the spectrum, and the correlation functions of basic physical quantities match exactly on both sides of the correspondence. In mathematical sense, this would constitute a proof of the equivalence of two theories. This is the spirit of

the celebrated Gubser-Klebanov-Polyakov-Witten (GKP-W) relation[2, 3] and it has been quite successful for the $\frac{1}{2}$ BPS quantities, largely because the supergravity approximation can be used on the string side.

To go beyond this approximation, the main difficulty resides in the extension of supergravity to incorporate the stringy excitations. The most direct way would be to construct a closed superstring field theory in the $AdS_5 \times S^5$ background, but it appears to be beyond reach at the present time. A more practical approach is to develop a worldsheet first quantized formalism and compute the correlation functions by constructing appropriate vertex operators anchored at the points on the boundary of AdS spacetime. Research in this direction was initiated in [12] using the Green-Schwarz formalism[13, 14] and subsequently in the pure spinor formalism[15]. Since then numerous investigations were made but most of them are classical or semi-classical and a full fledged quantization of a superstring (*i.e.* to all orders in α') in $AdS_5 \times S^5$ background has not been achieved. Consequently, the precise spectrum of the theory is not yet known. For recent reviews, readers are referred to [16, 17, 18] and references therein.

As a matter of fact, even a superparticle[19], which represents the zero mode of the superstring, has not been systematically quantized from the first principle in this curved background. We should note, however, that in a pioneering work [20] Metsaev wrote down a quadratic action for a light-cone superfield, which was invariant under a set of $psu(2, 2|4)$ generators made out of the coordinates and the momenta of a superparticle. Although the method was not systematic, this was equivalent to quantization of a superparticle. Concerning the spectrum of this system, some analysis of the AdS “mass” operator was performed but the AdS energy spectrum was not obtained. In subsequent developments[21, 22], the AdS energy was worked out for some subset of the states and was shown to agree with that of the corresponding supergravity fields. Also, advancements were made for the formalism itself, as a part of the formulation of the superstring. Classical action for a superstring in the light-cone gauge was derived explicitly based on the supercoset formalism in [23] and the construction of the generators of $psu(2, 2|4)$ was made more systematic in [24]. Nevertheless, these developments were purely classical.

In this article, we will be able to make substantial progress on the understanding of the quantum aspects of a Brink-Schwarz superparticle in $AdS_5 \times S^5$ with RR flux. It consists of (i) an exact systematic quantization from the first principle, including the derivation of the quantum Noether charges for the $psu(2, 2|4)$ (superconformal) symmetry, and (ii) complete solution of the spectrum of the theory with the explicit wave functions for the superconformal primaries. This is achieved in the physical light-cone gauge in the

phase space formulation. The spectrum agrees precisely with the supergravity results[25] [26], including all the Kaluza-Klein excitations. As a superparticle constitutes the zero-mode part of a superstring, our method and the result should shed light on the eventual quantization of a superstring in this important curved background.

We will now give the outline of our work, which at the same time serves to indicate the organization of the rest of this article. We will begin by describing, in section 2, the classical phase space formulation of a superparticle in the $AdS_5 \times S^5$ background. More specifically, after recalling the $psu(2, 2|4)$ symmetry algebra in section 2.1, we will review, in section 2.2, the supercoset method of constructing the invariant classical action in the “light-cone gauge”, first performed in [23]. Then we will develop the phase space formulation based on such an action in section 2.3. We will develop a powerful method of finding the Dirac brackets for the fundamental physical variables from the gauge-fixed action and find appropriate combinations which satisfy the canonical form of the bracket relations. The section 3 will be devoted to the construction of the quantum Noether charges for the $psu(2, 2|4)$ superconformal symmetry. We will first compute the Noether charges at the classical level in terms of the phase space variables and then quantize them by performing appropriate normal-ordering. All the quantum charges are explicitly obtained, which will be important in solving the system completely. In section 4, which is the main part of this article, we will give the complete solutions for the superconformal primary states of the system and show that the spectrum precisely agrees with the supergravity results. In preparation for the solution, we will first discuss, in section 4.1, the two choices of the scheme of the representation of the superconformal algebra, which will be called the dilatation (D) scheme and the energy (E) scheme. Then, after presenting the superconformal primary conditions in section 4.2, we will analyze the allowed highest weight unitary representations for the $su(4)$ sector in section 4.3. Finally in section 4.4, we will solve the superconformal primary conditions to obtain the wave functions explicitly and show that they enjoy expected properties. The section 5 is devoted to discussions and future perspectives. Several appendices are provided to display some further details.

2 Phase space formulation of a classical superparticle in $AdS_5 \times S^5$ with RR flux

We begin by describing the phase space formulation of a superparticle in $AdS_5 \times S^5$ background with RR flux at the classical level. We will adopt the the Brink-Green-Schwarz

formulation[19][13, 14] and basically follow the light-cone-gauge treatment of Metsaev and Tseytlin [12, 23] for a string in the above background. Upon dropping the dependence on σ , the coordinate along the string, we can specialize to the case of a particle. Therefore this section is mostly a review, except that a new important observation will be made in the subsection 2.3 concerning the systematic computation of the Dirac bracket.

2.1 $psu(2, 2|4)$ algebra in the light-cone basis

The most efficient way to construct the (Brink-)Green-Schwarz action for a string (and a particle) in $AdS_5 \times S^5$ background with RR flux is to make use of the supercoset method[27][12] based on the global symmetry group $PSU(2, 2|4)$, the bosonic part of which is $SO(4, 2) \times SO(6)$. Indeed, it is well-known that $AdS_5 \times S^5$ can be represented as the coset

$$AdS_5 \times S^5 \simeq \frac{SO(4, 2) \times SO(6)}{SO(4, 1) \times SO(5)}. \quad (2.1)$$

Therefore we must first discuss the generators of $PSU(2, 2|4)$, which form the Lie superalgebra $psu(2, 2|4)$.

The even part of $psu(2, 2|4)$ consists of $so(4, 2)$ and $so(6)$. $so(4, 2)$ can be regarded as acting on the six-dimensional flat space with coordinates $X^A = (X^{-1}, X^0, X^1, X^2, X^3, X^4)$ and the signature $(-, -, +, +, +, +)$. Its generators, to be denoted by T^{AB} , satisfy the commutation relations

$$[T^{AB}, T^{CD}] = \eta^{BC}T^{AD} - \eta^{AC}T^{BD} - \eta^{BD}T^{AC} + \eta^{AD}T^{BC}. \quad (2.2)$$

We adopt the convention that T^{AB} 's are anti-hermitian. T^{AB} can be decomposed with respect to the $so(4, 1)$ subalgebra, which will be denoted as

$$T^{AB} = (T^{\hat{a}\hat{b}}, T^{\hat{a}}). \quad (2.3)$$

Here $T^{\hat{a}} \equiv T^{\hat{a}, -1}$ and $\hat{a} = 0 \sim 4$. In the context of AdS/CFT, it will be useful to regard $SO(4, 2)$ as the conformal group in four dimensions. From this point of view, it is natural to introduce the ‘‘conformal basis’’ generators as¹

$$P^a \equiv \frac{1}{\sqrt{2}}(T^a - T^{a4}), \quad K^a \equiv -\frac{1}{\sqrt{2}}(T^a + T^{a4}), \quad (2.4)$$

$$D \equiv T^4, \quad J^{ab} \equiv T^{ab}, \quad (2.5)$$

¹Our definitions of D, P^a and K^a differ slightly from the ones used in [23]. In particular we take D to be opposite in sign because we prefer to have the momentum P^a to carry the dimension +1.

where P^a, K^a, D, J^{ab} are the generators of translation, the special conformal transformation, the dilatation and the Lorentz rotations respectively and the “Lorentz index” a runs over the range $0 \sim 3$. They satisfy the commutation relations

$$[J^{ab}, P^c] = \eta^{bc} P^a - \eta^{ac} P^b, \quad (2.6)$$

$$[J^{ab}, K^c] = \eta^{bc} K^a - \eta^{ac} K^b, \quad (2.7)$$

$$[P^a, K^b] = -\eta^{ab} D - J^{ab}, \quad (2.8)$$

$$[D, P^a] = P^a, \quad [D, K^a] = -K^a, \quad (2.9)$$

$$[J^{ab}, J^{cd}] = \eta^{bc} J^{ad} - \eta^{ac} J^{bd} - \eta^{bd} J^{ac} + \eta^{ad} J^{bc}. \quad (2.10)$$

In relation to the κ -symmetry gauge fixing, to be discussed later, we will often use the “light-cone basis” (in the sense of four dimensions). For the basic coordinates of the four dimensional space the light-cone components are defined as

$$x^\pm = \frac{1}{\sqrt{2}}(x^3 \pm x^0), \quad x = \frac{1}{\sqrt{2}}(x^1 + ix^2), \quad \bar{x} = \frac{1}{\sqrt{2}}(x^1 - ix^2). \quad (2.11)$$

In other words, the metric in this basis has non-vanishing components $\eta^{+-} = \eta^{-+} = 1$, $\eta^{x\bar{x}} = \eta^{\bar{x}x} = 1$. Accordingly, the generators of $so(4, 2)$ in this basis will be taken as

$$P^\pm, P^x, P^{\bar{x}}, K^\pm, K^x, K^{\bar{x}}, D, J^{+-}, J^{\pm x}, J^{\pm \bar{x}}, J^{x\bar{x}}. \quad (2.12)$$

Further we will employ the following simplified notations

$$P \equiv P^x, \bar{P} \equiv P^{\bar{x}}, K \equiv K^x, \bar{K} \equiv K^{\bar{x}}. \quad (2.13)$$

From these definitions it is straightforward to write down the commutation relations for the generators in the light-cone basis.

Next consider the remaining bosonic subalgebra $so(6)$. This will be interpreted as $su(4)$ since the fermionic generators of $psu(2, 2|4)$ transform under the fundamental and anti-fundamental representations of $su(4)$. The traceless $su(4)$ generators J^i_j ($i, j = 1 \sim 4$) satisfy the algebra

$$[J^i_j, J^k_n] = \delta^k_j J^i_n - \delta^i_n J^k_j. \quad (2.14)$$

Now we come to the odd part of the $psu(2, 2|4)$ algebra. It consists of 32 supercharges $Q^{\pm i}, Q_i^\pm, S^{\pm i}, S_i^\pm$ ($i = 1 \sim 4$), which transform, as said above, under $su(4)$ as

$$[J^i_j, Q_k^\pm] = -\delta^i_k Q_j^\pm + \frac{1}{4} \delta_j^i Q_k^\pm, \quad [J^i_j, Q^{\pm k}] = \delta_j^k Q^{\pm i} - \frac{1}{4} \delta_j^i Q^{\pm k}, \quad (2.15)$$

and similarly for the S supercharges. The superscripts \pm on Q and S generators indicate their charge with respect to the generator J^{+-} (*i.e.* the boost along the 3-direction), as the following commutation relations show:

$$[J^{+-}, Q^{\pm i}] = \pm \frac{1}{2} Q^{\pm i}, \quad [J^{+-}, Q_i^{\pm}] = \pm \frac{1}{2} Q_i^{\pm}, \quad (2.16)$$

$$[J^{+-}, S^{\pm i}] = \pm \frac{1}{2} S^{\pm i}, \quad [J^{+-}, S_i^{\pm}] = \pm \frac{1}{2} S_i^{\pm}. \quad (2.17)$$

We note that, together with the commutation relations for the bosonic generators already given, the values of the J^{+-} -charge for all the generators are in the finite range $[-1, +1]$. This fact will play an important role in the gauge fixing later.

The fermionic generators also carry charges with respect to the generators D and $J^{x\bar{x}}$. The charge assignment is expressed through the following commutation relations:

$$[D, Q^{\pm i}] = \frac{1}{2} Q^{\pm i}, \quad [D, Q_i^{\pm}] = \frac{1}{2} Q_i^{\pm}, \quad (2.18)$$

$$[D, S^{\pm i}] = -\frac{1}{2} S^{\pm i}, \quad [D, S_i^{\pm}] = -\frac{1}{2} S_i^{\pm}, \quad (2.19)$$

$$[J^{x\bar{x}}, Q^{\pm i}] = \pm \frac{1}{2} Q^{\pm i}, \quad [J^{x\bar{x}}, Q_i^{\pm}] = \mp \frac{1}{2} Q_i^{\pm}, \quad (2.20)$$

$$[J^{x\bar{x}}, S_i^{\pm}] = \pm \frac{1}{2} S_i^{\pm}, \quad [J^{x\bar{x}}, S^{\pm i}] = \mp \frac{1}{2} S^{\pm i}. \quad (2.21)$$

The transformation properties of the supercharges under the four dimensional Poincaré generators are as follows. Under the Lorentz rotations they transform as

$$[J^{+x}, Q^{-i}] = Q^{+i}, \quad [J^{+\bar{x}}, S^{-i}] = S^{+i}, \quad [J^{-\bar{x}}, Q^{+i}] = -Q^{-i}, \quad [J^{-x}, S^{+i}] = -S^{-i}, \quad (2.22)$$

$$[J^{+\bar{x}}, Q_i^{-}] = Q_i^{+}, \quad [J^{+x}, S_i^{-}] = S_i^{+}, \quad [J^{-x}, Q_i^{+}] = -Q_i^{-}, \quad [J^{-\bar{x}}, S_i^{+}] = -S_i^{-}, \quad (2.23)$$

while the commutation relations with the translation and the conformal boost generators take the form

$$[P^{\pm}, S^{\mp i}] = iQ^{\pm i}, \quad [\bar{P}, S^{-i}] = iQ^{-i}, \quad [P, S^{+i}] = -iQ^{+i}, \quad (2.24)$$

$$[P^{\pm}, S_i^{\mp}] = -iQ_i^{\pm}, \quad [P, S_i^{-}] = -iQ_i^{-}, \quad [\bar{P}, S_i^{+}] = iQ_i^{+}, \quad (2.25)$$

$$[K^{\pm}, Q^{\mp i}] = iS^{\pm i}, \quad [K, Q^{-i}] = iS^{-i}, \quad [\bar{K}, Q^{+i}] = -iS^{+i}, \quad (2.26)$$

$$[K^{\pm}, Q_i^{\mp}] = -iS_i^{\pm}, \quad [\bar{K}, Q_i^{-}] = -iS_i^{-}, \quad [K, Q_i^{+}] = iS_i^{+}. \quad (2.27)$$

Finally, the anticommutation relations between the supercharges are given by

$$\{Q^{\pm i}, Q_j^{\pm}\} = \mp i P^{\pm} \delta_j^i, \quad \{Q^{+i}, Q_j^{-}\} = -i P \delta_j^i, \quad \{Q_i^{+}, Q^{-j}\} = -i \bar{P} \delta_j^i, \quad (2.28)$$

$$\{S^{\pm i}, S_j^{\pm}\} = \pm i K^{\pm} \delta_j^i, \quad \{S^{-i}, S_j^{+}\} = i K \delta_j^i, \quad \{S_i^{-}, S^{+j}\} = i \bar{K} \delta_j^i, \quad (2.29)$$

$$\{Q^{+i}, S_j^+\} = -J^{+x}\delta_j^i, \quad \{Q_i^+, S^{+j}\} = J^{+\bar{x}}\delta_i^j, \quad (2.30)$$

$$\{Q^{-i}, S_j^-\} = -J^{-\bar{x}}\delta_j^i, \quad \{Q_i^-, S^{-j}\} = J^{-x}\delta_i^j, \quad (2.31)$$

$$\{Q^{\pm i}, S_j^\mp\} = \frac{1}{2}(J^{+-} + J^{x\bar{x}} \pm D)\delta_j^i \mp J^i_j, \quad (2.32)$$

$$\{Q_i^\pm, S^{\mp j}\} = \frac{1}{2}(-J^{+-} + J^{x\bar{x}} \mp D)\delta_i^j \mp J^j_i. \quad (2.33)$$

Hermiticity properties of the generators are such as to be consistent with the $psu(2, 2|4)$ algebra. Explicitly,

$$(P^\pm)^\dagger = -P^\pm, \quad P^\dagger = -\bar{P}, \quad (K^\pm)^\dagger = -K^\pm, \quad K^\dagger = -\bar{K}, \quad (2.34)$$

$$(J^{\pm x})^\dagger = -J^{\pm\bar{x}}, \quad (J^{+-})^\dagger = -J^{+-}, \quad (J^{x\bar{x}})^\dagger = J^{x\bar{x}}, \quad D^\dagger = -D, \quad (2.35)$$

$$(J^i_j)^\dagger = J^j_i, \quad (Q^{\pm i})^\dagger = Q_i^\pm, \quad (S^{\pm i})^\dagger = S_i^\pm. \quad (2.36)$$

This completes the description of the $psu(2, 2|4)$ algebra in the light-cone basis.

2.2 Supercoset construction

We are now ready to construct the action by the supercoset method. The supercoset of interest is $\mathcal{K} = \mathcal{G}/\mathcal{H}$ where

$$\mathcal{G} = PSU(2, 2|4), \quad \mathcal{H} = SO(4, 1) \times SO(5). \quad (2.37)$$

We follow [23] and take the representative element G of \mathcal{K} in the form

$$G = g_x g_\theta g_\eta g_\phi g_y, \quad (2.38)$$

$$g_x = \exp(x^a P_a), \quad (2.39)$$

$$g_\theta = \exp(\theta^{-i} Q_i^+ + \theta_i^- Q^{+i} + \theta^{+i} Q_i^- + \theta_i^+ Q^{-i}), \quad (2.40)$$

$$g_\eta = \exp(\eta^{-i} S_i^+ + \eta_i^- S^{+i} + \eta^{+i} S_i^- + \eta_i^+ S^{-i}), \quad (2.41)$$

$$g_\phi = \exp(\phi D), \quad (2.42)$$

$$g_y = \exp(y^i_j J^j_i), \quad y^i_j = \frac{i}{2}(\gamma^{A'})^i_j y^{A'}, \quad A' = 5 \sim 9. \quad (2.43)$$

The variables (x^a, ϕ) describe the AdS_5 part, while $y^{A'}$ are the coordinates of S^5 . The fermionic part of the coset is parametrized by the grassmann variables $\theta^{\pm i}, \theta_i^\pm, \eta^{\pm i}, \eta_i^\pm$, with the conjugation property $\theta_i^\pm = (\theta^{\pm i})^\dagger$, $\eta_i^\pm = (\eta^{\pm i})^\dagger$.

Perhaps a clarifying remark should be made on the choice of the coset parametrization, especially the part which is supposed to parametrize the AdS_5 , *i.e.* $g_{x,\phi} = \exp(x^a P_a) \exp(\phi D)$.

At first sight one might worry since the generators of $SO(4, 2)$ which do not appear in $g_{x,\phi}$ are $\{K^a, J^{ab}\}$ and they generate a group isomorphic to the Poincaré group in four dimensions (with K^a playing the role of the translation operators), which is not $SO(4, 1)$ but rather its contraction limit. Thus it would seem more legitimate to take $\exp(x^{\hat{a}}T_{\hat{a}})$ as the coset representative since we already saw in (2.3) that a natural decomposition of the generators of $SO(4, 2)$ into $SO(4, 1)$ and the coset part is given by $(T^{\hat{a}\hat{b}}, T^{\hat{a}})$. Actually, the choice of $g_{x,\phi}$ is perfectly legitimate. The reason is that an arbitrary element of $SO(4, 2)$ can be shown to be represented in the form $g_{x,\phi}h$, where $h = \exp(y_{\hat{a}\hat{b}}T^{\hat{a}\hat{b}}) \in SO(4, 1)$. (In fact we can use any embedding of $SO(4, 1)$ in $SO(4, 2)$ for this purpose.) All we have to make sure is that because (P_a, D) do not coincide with the coset directions $T^{\hat{a}}$ we must project out the motion along the coset manifold properly, as we will explain shortly.

As is well-known, the basic building block of the supercoset method is the Maurer-Cartan (MC) 1-form $J = G^{-1}dG$, which is invariant under the left action of $PSU(2, 2|4)$. As it takes its value in $psu(2, 2|4)$, it can be expanded as

$$\begin{aligned} J = G^{-1}dG &= L_P^a P_a + L_K^a K_a + L_D D + \frac{1}{2} L_J^{ab} J^{ab} + L^j{}_i J^i{}_j \\ &+ L_Q^{+k} Q_k^- + L_{Qk}^+ Q^{-k} + L_Q^{-k} Q_k^+ + L_{Qk}^- Q^{+k} \\ &+ L_S^{+k} S_k^- + L_{Sk}^+ S^{-k} + L_S^{-k} S_k^+ + L_{Sk}^- S^{+k}. \end{aligned} \quad (2.44)$$

In contrast to the case of the flat space time, all the generators, not just the coset generators, appear on the right hand side. This means that $G^{-1}dG$ as a whole describes the motion in the entire group space. What we really want is the motion along the bosonic coset, namely $AdS_5 \times S^5$. To extract this out, we need the orthogonal decomposition of the coset part and the rest. This is achieved by the use of the invariant bilinear form, commonly called the “supertrace”. It is given by [28]

$$\text{Str}(T^{\hat{a}}T^{\hat{b}}) = \eta^{\hat{a}\hat{b}}, \quad \text{Str}(T^{\hat{a}\hat{b}}T^{\hat{c}\hat{d}}) = \eta^{\hat{b}\hat{c}}\eta^{\hat{a}\hat{d}} - \eta^{\hat{a}\hat{c}}\eta^{\hat{b}\hat{d}}, \quad (2.45)$$

$$\text{Str}(J^i{}_j J^k{}_n) = -\delta_n^i \delta_j^k + \frac{1}{4} \delta_j^i \delta_n^k, \quad (2.46)$$

$$\text{Str}(Q^{\pm i} S_j^{\mp}) = \text{Str}(Q_i^{\mp} S^{\pm j}) = \pm \delta_j^i, \quad (2.47)$$

$$\text{Rest} = 0. \quad (2.48)$$

Concerning the $SO(4, 2)$ sector, we see from (2.45) that $T^{\hat{a}} = (T^a, T^4)$ are the desired coset generators which are orthogonal to the $so(4, 1)$ generators $T^{\hat{a}\hat{b}}$. As for the $SO(6)$ sector, we need to first convert the $SU(4)$ generators $J^i{}_j$ to $SO(6)$ generators and then decompose them into the $SO(6)/SO(5)$ part and the $SO(5)$ part. This is achieved with the aid of the 4×4 γ -matrices $\gamma^{A'}$ of $SO(5)$ and its antisymmetrized products $\gamma^{A'B'}$: If

we define

$$J^{A'} \equiv -\frac{i}{2} J^i_j (\gamma^{A'})^j_i, \quad J^{A'B'} \equiv -\frac{1}{2} J^i_j (\gamma^{A'B'})^j_i, \quad (2.49)$$

one can check that they together form the $so(6)$ algebra, where $J^{A'B'}$ generate the $so(5)$ and $J^{A'}$ represent the coset generators. Furthermore, from (2.46) one can easily obtain

$$\text{Str}(J^{A'} J^{B'}) = \delta^{A'B'}, \quad \text{Str}(J^{A'B'} J_{C'D'}) = \delta_{C'D'}^{A'B'}, \quad \text{Str}(J^{A'} J^{B'C'}) = 0, \quad (2.50)$$

showing that the $J^{A'}$ form the desired orthogonal basis of the coset $SO(6)/SO(5)$. Therefore, we should rewrite the MC 1-form (2.44) as $J = J_B + \text{rest}$ and extract the bosonic coset part J_B defined by

$$J_B \equiv J_B^{\hat{a}} T_{\hat{a}} + J_B^{A'} J_{A'}. \quad (2.51)$$

From the definitions (2.4) and (2.49) we find

$$J_B^{\hat{a}} = (J_B^a, J_B^A) = \left(\frac{1}{\sqrt{2}} (L_P^a - L_K^a), L_D \right), \quad J_B^{A'} = \frac{i}{2} (\gamma^{A'})^i_j L^j_i. \quad (2.52)$$

Then the Lagrangian of a superparticle in $AdS_5 \times S^5$ is given by

$$L = \frac{1}{2e} \text{Str}(J_B J_B) = \frac{1}{2e} \left(\eta_{\hat{a}\hat{b}} J_B^{\hat{a}} J_B^{\hat{b}} + \delta_{A'B'} J_B^{A'} J_B^{B'} \right), \quad (2.53)$$

where e is the einbein and we have used the same symbol J_B to mean the coefficient of $d\tau$ in the 1-form J_B , where τ is the parameter along the worldline. Note that for a superparticle the Wess-Zumino term which is crucial for the κ -invariance in the superstring case vanishes since it contains a derivative with respect to σ . Indeed, the action above already possesses the desired κ symmetry.

Although the Lagrangian above has the virtue of being manifestly invariant under the $psu(2, 2|4)$ symmetry, it cannot be computed explicitly. The reason is that the MC 1-form $G^{-1}dG$ can contain up to 32 powers of fermionic coordinates and it is practically impossible to compute it in closed form. This problem can be solved by imposing judicious gauge conditions. A convenient set of 16 conditions we adopt are the so-called semi-light-cone gauge conditions given by

$$\theta^{+i} = \theta_i^+ = \eta^{+i} = \eta_i^+ = 0, \quad (2.54)$$

which will often be denoted simply as $\Theta_I^+ = 0$. This means that only the supercharges with the J^{+-} charge $+\frac{1}{2}$ are kept in the coset representative G . Consequently, g_θ and g_η are reduced to

$$g_\theta = \exp(\theta^i Q_i^+ + \theta_i Q^{+i}), \quad g_\eta = \exp(\eta^i S_i^+ + \eta_i S^{+i}), \quad (2.55)$$

$$\theta^i \equiv \theta^{-i}, \quad \theta_i \equiv \theta_i^-, \quad \eta^i \equiv \eta^{-i}, \quad \eta_i \equiv \eta_i^-. \quad (2.56)$$

Here and hereafter, we suppress the superscript “−” for the the remaining fermionic coordinates for simplicity.

In this gauge, because the maximum value of the J^{+-} charge for the $psu(2, 2|4)$ generators is +1, the expansion of the MC 1-form J in powers of θ and η terminates in a few steps and one obtains simple explicit expressions for the components of J . The ones needed to construct the action take the form [23]

$$L_{P^+} = e^{-\phi} dx^+, \quad L_{P^-} = e^{-\phi} \left(dx^- - \frac{i}{2} \tilde{\theta}^i d\tilde{\theta}_i - \frac{i}{2} \tilde{\theta}_i d\tilde{\theta}^i \right), \quad (2.57)$$

$$L_{P^x} = e^{-\phi} dx, \quad L_{P^{\bar{x}}} = e^{-\phi} d\bar{x}, \quad (2.58)$$

$$L_{K^+} = L_{K^x} = L_{K^{\bar{x}}} = 0, \quad L_{K^-} = e^{\phi} \left(\frac{1}{4} (\tilde{\eta}^2)^2 dx^+ + \frac{i}{2} \tilde{\eta}^i d\tilde{\eta}_i + \frac{i}{2} \tilde{\eta}_i d\tilde{\eta}^i \right), \quad (2.59)$$

$$L_D = d\phi, \quad L^i_j = (dUU^{-1})^i_j + i \left(\tilde{\eta}^i \tilde{\eta}_j - \frac{1}{4} \tilde{\eta}^2 \delta^i_j \right) dx^+. \quad (2.60)$$

Here the matrix U is given by $U = \exp((i/2)y^{A'}\gamma^{A'})$ and the tilded fermionic variables are defined as

$$\tilde{\theta}^i \equiv U^i_j \theta^j, \quad \tilde{\theta}_i \equiv \theta_j (U^{-1})^j_i, \quad d\tilde{\theta}^i \equiv U^i_j d\theta^j, \quad d\tilde{\theta}_i \equiv d\theta_j (U^{-1})^j_i, \quad (2.61)$$

and similarly for $\tilde{\eta}$'s. (More explicit form of U is displayed in the Appendix A.) Substituting these expressions into (2.53) the action is easily obtained as

$$\begin{aligned} S = \int d\tau \frac{1}{2e} & \left(e^{-2\phi} \left(\dot{x}^+ \dot{x}^- + \dot{x} \dot{\bar{x}} + e^{2\phi} (\dot{\phi})^2 \right) + (e_0^{A'})^2 \right. \\ & - \frac{i}{2} \dot{x}^+ \left[e^{-2\phi} (\theta^i \dot{\theta}_i + \theta_i \dot{\theta}^i) + \eta^i \dot{\eta}_i + \eta_i \dot{\eta}^i - 2ie_0^{A'} \tilde{\eta}_i (\gamma^{A'})^i_j \tilde{\eta}^j \right] \\ & \left. - \frac{1}{4} (\dot{x}^+)^2 \left[(\eta^2)^2 - (\tilde{\eta}_i (\gamma^{A'})^i_j \tilde{\eta}^j)^2 \right] \right), \end{aligned} \quad (2.62)$$

where $e_0^{A'} = -\frac{i}{2} \text{Tr}(\gamma^{A'} \dot{U} U^{-1})$. Note that if we define a variable z by $z \equiv e^{\phi}$, the first three terms can be rewritten as

$$\frac{1}{z^2} \left[\frac{1}{2} \left(\frac{dx^a}{d\tau} \right)^2 + \left(\frac{dz}{d\tau} \right)^2 \right]. \quad (2.63)$$

This shows that the present parametrization of the coset corresponds to the familiar Poincaré coordinates for the AdS_5 part, up to a trivial scaling.

2.3 Classical phase space formulation

In the preceding subsection we reviewed the construction of the gauge-fixed action for a superparticle in the $AdS_5 \times S^5$ background in the configuration space. With the use of

the light-cone gauge the form of the action has been simplified substantially. Nevertheless it is still quite non-linear and it is difficult to obtain the general solutions of the equations of motion, which are needed for the canonical quantization procedure.

In such a situation the phase space formulation can be quite powerful. In particular, when the generator of the dynamics is contained in the symmetry algebra, we may first perform the quantization at equal time without solving the dynamical equations of motion and then generate the dynamics algebraically by a member of the algebra². This applies to the present case, where the generators relevant for the dynamics, namely the AdS energy operator $E = -i(P^0 - K^0)/\sqrt{2}$ and the light-cone Hamiltonian operator $-P^-$, are in the $psu(2,2|4)$ algebra. For this reason we will develop the phase space formulation for our system in this subsection first at the classical level. In the next section we will perform the quantization and construct the quantum Noether charges which generate the $psu(2,2|4)$ algebra.

Although the general procedure for the phase space formulation is a textbook matter, it is not so easy to execute it in the present case because we do not have the explicit form of the un-gauge-fixed action: All we have is the action on the gauge slice $\Theta_I^+ = 0$. In fact we face a trouble right from the beginning since obviously the momenta $(P_{\theta^+}, P_{\eta^+})$ conjugate to these variables cannot be computed. To avoid this problem, one needs to compute the action at least up to first order in $\dot{\Theta}_I^+$, where Θ_I^+ denotes (θ^+, η^+) collectively. Suppose we have obtained such an action with additional efforts. Then we can compute the momenta, define the Poisson brackets for the basic phase space variables, and find all the constraints à la Dirac. Let us focus among them on the first class fermionic constraints expressing the κ symmetry and denote them by $\Phi_J = 0$. To fix the gauge by the conditions $\Theta_I^+ = 0$ and compute the Dirac bracket, we need to know the knowledge of the Poisson brackets among the constraints, including the gauge fixing conditions. Since Φ_J can contain the fermionic momenta $P_{\Theta^+} = \partial L / \partial \dot{\Theta}^+$ as well as Θ^+ variables, this computation actually requires the knowledge of the action to order Θ^+ and $\Theta^+ \dot{\Theta}^+$ as well.

Summarizing, to execute the usual procedure for the phase space formulation of our system of interest, we need to know the action not only on the gauge-slice but also slightly away from it, to order Θ^+ , $\dot{\Theta}^+$ and $\Theta^+ \dot{\Theta}^+$. Because of this reason, logically satisfactory derivation of the Dirac brackets has not been performed in the past based on the gauge-fixed action³.

²This feature was emphasized and utilized in [29] in the analysis of the superstring in the plane-wave background in the semi-light-cone conformal gauge.

³In [24] the Dirac brackets were derived by applying the usual Dirac's method directly to the light-cone-gauge-fixed action. Although this turned out to yield the correct brackets for this system, it is not guaranteed to be a legitimate procedure in general. The reason is as follows: To compute the Dirac bracket

We now make an important observation that, despite the apparent lack of the necessary information, there is in fact a systematic way to compute the Dirac brackets using only the knowledge of the gauge-fixed form of G and $G^{-1}dG$, with a small assumption which will be a posteriori justified. The basic idea is that, instead of computing the Dirac bracket directly, we will derive the general formula for the Lagrange bracket, which is the inverse of the Dirac bracket. Then we recognize that for the “physical” variables, *i.e.* the variables other than Θ^+ , the formula for the Lagrange bracket does not contain the derivatives with respect to Θ^+ and $\dot{\Theta}^+$. This means that the Lagrange brackets among the physical variables, which form a matrix, can be computed on the gauge slice. Furthermore, we find that this matrix is invertible, indicating that the choice of the gauge $\Theta^+ = 0$ is a proper one, and this inverse gives the Dirac brackets for the physical variables we want.

To make the logic clear, we shall demonstrate this in slightly abstract notations and then apply the formulas to our specific system to give concrete results. In the following, we collectively denote the bosonic and the fermionic variables by $X^{\bar{a}}$ and $\Theta^{\bar{\alpha}}$ respectively and write our Lagrangian as

$$L = \frac{1}{2e} J^{\bar{a}} J^{\bar{a}}. \quad (2.64)$$

Here $J^{\bar{a}}$ represent the components of the appropriate currents along the bosonic coset space, *not yet gauge-fixed*. $J^{\bar{a}}$ is utmost linear in \dot{X} or $\dot{\Theta}$. The momenta conjugate to $X^{\bar{a}}$ and $\Theta^{\bar{\alpha}}$ are given by

$$P_{\bar{a}} = \frac{\partial L}{\partial \dot{X}^{\bar{a}}} = \frac{1}{e} \frac{\partial J^{\bar{b}}}{\partial \dot{X}^{\bar{a}}} J^{\bar{b}}, \quad (2.65)$$

$$P_{\bar{\alpha}} = \frac{\partial L}{\partial \dot{\Theta}^{\bar{\alpha}}} = \frac{1}{e} \frac{\partial J^{\bar{b}}}{\partial \dot{\Theta}^{\bar{\alpha}}} J^{\bar{b}}. \quad (2.66)$$

As is true for our system, we consider the case where the matrix $M_{\bar{a}}^{\bar{b}} \equiv \partial J^{\bar{b}} / \partial \dot{X}^{\bar{a}}$ is invertible. Then from (2.65) we can solve for $J_{\bar{b}}$ as

$$J_{\bar{b}} = e(M^{-1})_{\bar{b}}^{\bar{a}} P_{\bar{a}}. \quad (2.67)$$

Putting this into (2.66) we obtain $d_{\bar{\alpha}} = 0$ where

$$d_{\bar{\alpha}} = P_{\bar{\alpha}} - \frac{\partial J^{\bar{b}}}{\partial \dot{\Theta}^{\bar{\alpha}}} (M^{-1})_{\bar{b}}^{\bar{a}} P_{\bar{a}}. \quad (2.68)$$

one needs the knowledge of the matrix D formed by the Poisson brackets among the constraints. Let us denote by D_g and D_{ung} such matrices obtained from the gauge-fixed and the un-gauge-fixed actions respectively. Clearly D_g is a submatrix of D_{ung} because there are less constraints for the gauge-fixed theory. Now in order for the procedure starting from the gauge-fixed action to yield the correct Dirac bracket for the physical variables, the inverse D_g^{-1} must be realized as a block submatrix in D_{ung}^{-1} . This however is not necessarily true and has to be checked.

As $d_{\bar{\alpha}}$'s consist of basic phase space variables only, they represent fermionic constraints. (There are also bosonic constraints generated by the presence of the einbein, but as they are not important in the ongoing analysis, we will discuss them later.) Now we make an assumption that (2.68) are the only fermionic constraints and that the κ -gauge symmetry generated by half of them can be fixed by setting $\Theta_I^+ = 0$, where Θ_I^+ represents an appropriate half of $\Theta^{\bar{\alpha}}$. The remaining “physical” part will be denoted by Θ_I^- . This assumption is quite reasonable since the degrees of freedom of the system should not differ from the flat case. In any case, it will be supported by the results of our analysis.

We now wish to compute the Lagrange bracket among the physical phase space variables $(X^{\bar{a}}, P_{\bar{a}}, \Theta_I^-)$. Let us first give a brief review of this bracket for the case without constraints. Let (p_i, q^i) be a basis of the $2N$ dimensional phase space, including fermionic variables. As we have used left derivative to define the fermionic momenta, the appropriate definition of the Poisson bracket for arbitrary functions F and G is

$$\{F, G\}_P = (-1)^{|i|} \frac{\partial F}{\partial_R q^i} \frac{\partial G}{\partial_L p_i} - \frac{\partial F}{\partial_R p_i} \frac{\partial G}{\partial_L q^i}. \quad (2.69)$$

Here the subscripts L and R refer to the left and the right derivatives respectively and $|i| = 0(1)$ for the bosonic (fermionic) variable. Let $\{z_\mu\}_{\mu=1 \sim 2N}$ be functions of (p, q) which form a complete basis of the phase space. Then, the Lagrange bracket between z_μ and z_ν is given by⁴

$$(z_\mu, z_\nu)_L = (-1)^{|i|} \frac{\partial p^i}{\partial_L z_\nu} \frac{\partial q_i}{\partial_R z_\mu} - \frac{\partial q_i}{\partial_L z_\nu} \frac{\partial p^i}{\partial_R z_\mu}. \quad (2.70)$$

One can easily show that the Lagrange bracket is the inverse of the Poisson bracket in the sense

$$(z_\mu, z_\nu)_L \{z_\mu, z_\rho\}_P = \delta_{\nu\rho}. \quad (2.71)$$

Next consider the case with constraints. As said before we assume that by adding the gauge-fixing constraints $\Theta_I^+ = 0$, the total set of M constraints $(d_{\bar{\alpha}}, \Theta_I^+)$ can be made second class. It can then be shown that if we take $(d_{\bar{\alpha}}, \Theta^+)$ themselves to be among the z_μ functions, the counter part of the relation (2.71) holds for the $2N - M$ physical variables in the form

$$\sum_{\bar{\mu}=1}^{2N-M} (z_{\bar{\mu}}, z_{\bar{\nu}})_L \{z_{\bar{\mu}}, z_{\bar{\rho}}\}_D = \delta_{\bar{\nu}\bar{\rho}}, \quad (\bar{\nu}, \bar{\rho} = 1, \dots, 2N - M), \quad (2.72)$$

⁴It should be clear that the subscript L on the bracket stands for “Lagrange” and not for “Left”.

where $\{z_{\bar{\mu}}, z_{\bar{\nu}}\}_D$ is the Dirac bracket. This means that the Dirac bracket for the physical variables can be computed as the inverse of the their Lagrange bracket.

Let us compute the Lagrange brackets $(z_{\bar{\mu}}, z_{\bar{\nu}})_L$ more explicitly by taking

$$(p_i, q^i) = (X^{\bar{a}}, P_{\bar{a}}, \Theta_I^-, \Theta_I^+, P_{\bar{\alpha}}), \quad (2.73)$$

$$(z_{\bar{\mu}}) = (z_{\bar{\mu}}, \Theta_I^+, d_{\bar{\alpha}}), \quad z_{\bar{\mu}} = (X^{\bar{a}}, P_{\bar{a}}, \Theta_I^-), \quad (2.74)$$

Note that (p, q) and $(z_{\bar{\mu}})$ differ only by $P_{\bar{\alpha}} \leftrightarrow d_{\bar{\alpha}}$ and $P_{\bar{\alpha}}$ can be regarded as a function of $z_{\bar{\mu}}$ by the use of the relation (2.68), namely $P_{\bar{\alpha}} = d_{\bar{\alpha}} + (\partial J^{\bar{b}} / \partial \dot{\Theta}^{\bar{\alpha}}) M_{\bar{b}}^{-1\bar{a}} P_{\bar{a}}$. Although the general definition of the Lagrange bracket is already given in (2.70), let us display it again for the physical variables $z_{\bar{\mu}}$, as it will be very important:

$$(z_{\bar{\mu}}, z_{\bar{\nu}})_L = (-1)^{|i|} \frac{\partial p^i}{\partial_L z_{\bar{\nu}}} \frac{\partial q_i}{\partial_R z_{\bar{\mu}}} - \frac{\partial q_i}{\partial_L z_{\bar{\nu}}} \frac{\partial p^i}{\partial_R z_{\bar{\mu}}}. \quad (2.75)$$

We now make two simple but crucial observations about this formula. First, since Θ^+ is not among the variables $z_{\bar{\mu}}$ the derivative with respect to Θ^+ cannot appear on the right hand side. Second, its conjugate P_{Θ^+} can only appear with Θ^+ in the form like $(\partial P_{\Theta^+} / \partial z_{\bar{\mu}})(\partial \Theta^+ / \partial z_{\bar{\nu}})$. But this vanishes because $\partial \Theta^+ / \partial z_{\bar{\nu}} = 0$. Hence $P_{\Theta^+} = \partial L / \partial \dot{\Theta}^+$ never appears in (2.75). Combining, we find that the Lagrange bracket for the physical variables can be computed without knowing the dependence on Θ^+ and $\dot{\Theta}^+$. In other words, the knowledge of the relevant quantities *on the gauge slice* is sufficient to compute it. It is straightforward to evaluate the right hand side of (2.75) for the choice (2.73) and (2.74) and obtain the following useful formulas:

$$(X^{\bar{a}}, X^{\bar{b}})_L = (P_{\bar{a}}, P_{\bar{b}})_L = 0, \quad (X^{\bar{a}}, P_{\bar{b}})_L = \delta_{\bar{b}}^{\bar{a}}, \quad (2.76)$$

$$(X^{\bar{a}}, \Theta_I^-)_L = - \left(\frac{\partial P_{\Theta_I^-}}{\partial X^{\bar{a}}} \right)_{\Theta^+=0}, \quad (P_{\bar{a}}, \Theta_I^-)_L = - \left(\frac{\partial P_{\Theta_I^-}}{\partial P_{\bar{a}}} \right)_{\Theta^+=0}, \quad (2.77)$$

$$(\Theta_I^-, \Theta_J^-)_L = - \left(\frac{\partial P_{\Theta_I^-}}{\partial_L (\Theta_J^-)} + \frac{\partial P_{\Theta_J^-}}{\partial_R (\Theta_I^-)} \right)_{\Theta^+=0}. \quad (2.78)$$

Having explained our method of computation, let us apply it to the superparticle case at hand. From the explicit form of the MC 1-forms given in (2.57) \sim (2.60), it is straightforward to compute the fermionic momenta on the gauge slice, namely $P_{\bar{\alpha}} = (\partial J^{\bar{b}} / \partial \dot{\Theta}^{\bar{\alpha}}) (M^{-1})_{\bar{b}}^{\bar{a}} P_{\bar{a}} \Big|_{\Theta^+=0}$. The result is

$$P_{\theta^i} = \frac{i}{2} \theta_i^- P_-, \quad P_{\theta^i} = \frac{i}{2} \theta^i P_-, \quad (2.79)$$

$$P_{\eta^i} = \frac{i}{2} \eta_i e^{2\phi} P_-, \quad P_{\eta^i} = \frac{i}{2} \eta^i e^{2\phi} P_-. \quad (2.80)$$

Substituting them into the general formulas (2.76) \sim (2.78), we readily obtain the explicit form of the Lagrange brackets. In the formulas below, we use the notation $x^{\underline{a}}$ to represent the set of bosonic coordinates $(x^+, x^-, x, \bar{x}, \phi(=x^4), y^{A'})$. Then the results can be written as

$$(x^{\underline{a}}, x^{\underline{b}})_L = (P_{\underline{a}}, P_{\underline{b}})_L = 0, \quad (x^{\underline{a}}, P_{\underline{b}})_L = \delta_{\underline{b}}^{\underline{a}} \quad (2.81)$$

$$(x^{\underline{a}}, \theta^i)_L = (x^{\underline{a}}, \theta_i)_L = 0, \quad (2.82)$$

$$(x^{\underline{a}}, \eta^i)_L = -i\eta_i e^{2\phi} P_- \delta_{\underline{a}}^i, \quad (x^{\underline{a}}, \eta_i)_L = -i\eta^i e^{2\phi} P_- \delta_{\underline{a}}^i, \quad (2.83)$$

$$(P_{\underline{a}}, \theta^i)_L = -\frac{i}{2}\theta_i \delta_{\underline{a}}^i, \quad (P_{\underline{a}}, \theta_i)_L = -\frac{i}{2}\theta^i \delta_{\underline{a}}^i, \quad (2.84)$$

$$(P_{\underline{a}}, \eta^i)_L = -\frac{i}{2}\eta_i e^{2\phi} \delta_{\underline{a}}^i, \quad (P_{\underline{a}}, \eta_i)_L = -\frac{i}{2}\eta^i e^{2\phi} \delta_{\underline{a}}^i, \quad (2.85)$$

$$(\theta^i, \theta_j)_L = -iP_- \delta_j^i, \quad (\eta^i, \eta_j)_L = -ie^{2\phi} P_- \delta_j^i. \quad (2.86)$$

From these expressions we confirm that the Lagrange bracket $(z_{\bar{\mu}}, z_{\bar{\nu}})_L$ as a matrix is invertible, justifying our assumption made earlier. Actual inversion is quite easy and we obtain the Dirac brackets as

$$\{x^{\underline{a}}, P_{\underline{b}}\}_D = \delta_{\underline{b}}^{\underline{a}}, \quad (2.87)$$

$$\{x^{\underline{a}}, \theta^i\}_D = -\frac{1}{2P_-} \theta^i \delta_{\underline{a}}^i, \quad \{x^{\underline{a}}, \theta_i\}_D = -\frac{1}{2P_-} \theta_i \delta_{\underline{a}}^i, \quad (2.88)$$

$$\{x^{\underline{a}}, \eta^i\}_D = -\frac{1}{2P_-} \eta^i \delta_{\underline{a}}^i, \quad \{x^{\underline{a}}, \eta_i\}_D = -\frac{1}{2P_-} \eta_i \delta_{\underline{a}}^i, \quad (2.89)$$

$$\{P_{\underline{a}}, \eta^i\}_D = \eta^i \delta_{\underline{a}}^i, \quad \{P_{\underline{a}}, \eta_i\}_D = \eta_i \delta_{\underline{a}}^i, \quad (2.90)$$

$$\{\theta^i, \theta_j\}_D = \frac{i}{P_-} \delta_j^i, \quad \{\eta^i, \eta_j\}_D = \frac{i}{P_-} e^{-2\phi} \delta_j^i. \quad (2.91)$$

It is evident that, just as in the flat case, the variables θ and η no longer satisfy the canonical bracket relations. The experience for the flat case suggests that we should form the following combinations:

$$S_{\theta^i} = \sqrt{P_-} \theta^i, \quad S_{\theta_i} = \sqrt{P_-} \theta_i, \quad S_{\eta^i} = \sqrt{P_-} e^{\phi} \eta^i, \quad S_{\eta_i} = \sqrt{P_-} e^{\phi} \eta_i. \quad (2.92)$$

The extra factor e^{ϕ} is introduced for the η 's to make their conformal weight equal to that of θ 's. Then it is not difficult to check that these new variables commute with the bosonic variables and satisfy the canonical bracket relations

$$\{S_{\theta^i}, S_{\theta_j}\}_D = \{S_{\eta^i}, S_{\eta_j}\}_D = i\delta_j^i, \quad \text{Rest} = 0. \quad (2.93)$$

Finally, let us discuss the remaining first class bosonic constraints which so far have been suppressed. They are $P_e = 0, T = 0$, where P_e is the momentum conjugate to the

einbein $e(\tau)$ and T is the reparametrization generator. T is related to the Hamiltonian H by $eT = H$. The calculation of the canonical Hamiltonian in the semi-light-cone gauge is straightforward but slightly cumbersome. Since all the terms in the action (2.53) are quadratic in the time derivative, we have

$$H = L = \frac{1}{2e} \left(2J_B^+ J_B^- + 2J_B^x J_B^{\bar{x}} + (J_B^\phi)^2 + J_B^{A'} J_B^{A'} \right). \quad (2.94)$$

What is non-trivial is the step of expressing the relevant MC 1-forms in terms of the phase space variables. This requires the explicit evaluation of the formula (2.67), with the aid of some formulas of the Appendix A. After some computations we get

$$H = e \left(2e^{2\phi} (P_+ P_- + P_x P_{\bar{x}}) + \frac{1}{2} (P_\phi^2 + \hat{l}^2 + (S_\eta^2)^2) - 2S_{\eta_i} l^i_j S_{\eta^j} \right). \quad (2.95)$$

Here l^i_j and \hat{l}^2 are, respectively, the orbital part of the $su(4)$ generator and the associated quadratic Casimir operator, which are discussed in the Appendix A. As our primary goal in this paper is to compute the physical spectrum of the system, we will fix the gauge symmetries generated by these constraints as well by imposing the conditions

$$\xi(\tau) = e(\tau) - 1, \quad \chi(\tau) = x^+(\tau) - \tau. \quad (2.96)$$

In this bonafide light-cone gauge, all the constraints become second class. The addition of the conditions above requires us to modify the Dirac bracket slightly. However, it is easy to see that only the brackets with P_+ need to be changed and its effect can be implemented within the unmodified Dirac bracket by replacing P_+ by the expression

$$P_+ = -\frac{P_x P_{\bar{x}}}{P_-} - \frac{e^{-2\phi}}{4P_-} \left(P_\phi^2 + (\hat{l}^2) + (S_\eta^2)^2 \right) + e^{-2\phi} S_{\eta_i} l^i_j S_{\eta^j}, \quad (2.97)$$

which is obtained by solving the constraint $T = 0$ explicitly. With this understanding we need not modify our Dirac bracket. Due to the gauge condition (2.96) $x^+(= \tau)$ becomes non-dynamical and the τ -evolution of any function F is generated by the light-cone Hamiltonian $H^{l.c.} = -P_+$ as $dF/d\tau = \partial F/\partial\tau + \{F, H^{l.c.}\}_D$.

3 Quantization and the quantum Noether charges

3.1 Quantization

Now that we have clarified the phase space formulation of the dynamics of a superparticle in $AdS_5 \times S^5$ and obtained the Dirac brackets, it is straightforward to quantize our system: We simply replace $i\{, \}_D$ by the equal time quantum commutator $[,]$. In addition, for

convenience we will introduce simplified notations for the quantized fermionic variables. The new variables $S^i, S_i, \tilde{S}^i, \tilde{S}_i$ are defined as

$$S^i = iS_{\theta^i}, \quad S_i = iS_{\theta_i}, \quad \tilde{S}^i = iS_{\eta^i}, \quad \tilde{S}_i = iS_{\eta_i}. \quad (3.1)$$

Then the commutation relations of the fundamental variables take the form

$$[x, P_x] = [\bar{x}, P_{\bar{x}}] = [x^-, P_-] = [\phi, P_\phi] = i, \quad [y^{A'}, P_{B'}] = i\delta_{B'}^{A'}, \quad (3.2)$$

$$\{S^i, S_j\} = \{\tilde{S}^i, \tilde{S}_j\} = \delta_j^i, \quad \text{Rest} = 0. \quad (3.3)$$

3.2 Derivation of the Noether charges and their quantization

As explained at the beginning of section 2.3, our strategy for the solution of the quantum dynamics of a superparticle is to make use of the realization of the $psu(2, 2|4)$ symmetry of the system. In preparation for this goal, we shall derive in this subsection the Noether charges for this symmetry and quantize them in a systematic manner.

To begin, let us first recall how we can find these charges by the Noether method in the configuration space. Before gauge fixing, the $PSU(2, 2|4)$ transformation acts on the supercoset representative G in the manner

$$G \longrightarrow f_0 G h, \quad (3.4)$$

where $f_0 = e^{\epsilon_0}$ is an element of $PSU(2, 2|4)$ and h is a compensating $SO(4, 1) \times SO(5)$ transformation which depends on X, Θ and f_0 . For a global ϵ_0 the action is invariant under this transformation. To derive the Noether charges, one makes the parameter local and replaces f_0 by $f = e^{\epsilon(\tau)}$. Then, the bosonic coset part of the MC 1 form J_B gets transformed as

$$\begin{aligned} J_B &\longrightarrow [(fGh)^{-1}d(fGh)]_B = [h^{-1}G^{-1}(f^{-1}df)Gh + h^{-1}G^{-1}dGh + h^{-1}dh]_B \\ &= [G^{-1}\dot{\epsilon}^{\hat{A}}T_{\hat{A}}G]_B + \dots, \end{aligned} \quad (3.5)$$

where $T_{\hat{A}}$ denotes the generator of $psu(2, 2|4)$ and the ellipses stand for terms independent of $\dot{\epsilon}$ or higher order in ϵ . From this one can obtain the Noether charge $\mathbb{Q}_{\hat{A}}$ corresponding to $T_{\hat{A}}$ as⁵

$$\begin{aligned} \mathbb{Q}_{\hat{A}} &= \frac{1}{e\dot{\epsilon}} \text{Str}(\delta_{\hat{A}}J)_B J_B = \frac{1}{e} (G^{-1}T_{\hat{A}}G)^a J_B^a \\ &= \frac{1}{e} \left(\frac{1}{\sqrt{2}} (-(\delta_{\hat{A}}L_K^a) + (\delta_{\hat{A}}L_P^a)) J_B^a + (\delta_{\hat{A}}L_B^D) J_B^4 + \frac{i}{2} (\delta_{\hat{A}}L_B)^j{}_i (\gamma^{A'})^i{}_j J_B^{A'} \right). \end{aligned} \quad (3.6)$$

⁵We will denote the Noether charges in bold blackboard style, to distinguish them from the generic $psu(2, 2|4)$ generators.

In this formula, $\delta_{\hat{A}}L$'s are the expressions which appear in the expansion

$$G^{-1}T_{\hat{A}}G = (\delta_{\hat{A}}L_K^a)K^a + (\delta_{\hat{A}}L_P^a)P^a + \dots \quad (3.7)$$

Now when one fixes the gauge, the naive transformation law is no longer valid. Since the $PSU(2, 2|4)$ transformations in general do not preserve the gauge, one must perform appropriate compensating gauge transformations in order to keep the gauge condition intact. For a superstring in the $AdS_5 \times S^5$ background, the cumbersome task of finding such transformations was accomplished in [30].

Next let us discuss the case of the phase space formulation. Compared to the procedure in the configuration space just reviewed, the computation in the phase space formulation is much simpler. In particular, we need not find the compensating transformations explicitly once we have the proper Dirac bracket. This is because the Dirac bracket, by definition, automatically provides the requisite projection onto the gauge slice. Moreover, it solves another related problem at the same time. This is the apparent problem of ambiguities one encounters when one tries to convert the configuration space expressions into those in the phase space. Namely, any combination of the constraints can be added in the conversion formula. It should however be clear that as far as the computations using the Dirac brackets are concerned this is of no problem. Under the Dirac bracket, the constraints can be set strongly to zero and the result is unambiguous. Therefore, the formula (3.6) can be used as it is, with the replacement $J_{B\underline{a}} = e((\partial J/\partial \dot{X})^{-1})_{\underline{a}}^{\underline{b}}P_{\underline{b}}$. Hence, we have the formula

$$\mathbb{Q}_{\hat{A}} = (G^{-1}T_{\hat{A}}G)^{\underline{a}}((\partial J/\partial \dot{X})^{-1})_{\underline{a}}^{\underline{b}}P_{\underline{b}}, \quad (3.8)$$

which can be evaluated directly on the gauge slice. After involved but straightforward calculations, we obtain all the classical Noether charges and check that they satisfy the $psu(2, 2|4)$ algebra under the Dirac bracket. As we shortly display the quantum version of the Noether charges in full detail, the list of the classical charges so obtained is relegated to the Appendix B to avoid redundancy⁶.

The remaining problem is to find the quantum representation of the charges. The main task is to fix the ordering of the operators and it can be done by requiring the realization of the hermiticity properties (2.36) and the closure of the $psu(2, 2|4)$ algebra. One simplifying fact is that the τ -dependence is generated by a member of the algebra,

⁶In [24] classical Noether charges for the superstring case were obtained (although some of them were not displayed explicitly). When restricted to the superparticle mode, they agree with our results. However, in contrast to our direct systematic derivation, they resorted to some indirect reasoning for obtaining the dynamical generators.

namely by $H^{l.c.} = -P_+$. Thus, we can work at the time slice $\tau = 0$ and later recover the τ -dependence.

The first step is to impose the Hermiticity conditions (2.36) on the Noether charges. The rules of conjugation for the basic variables are

$$(x^{\underline{a}})^\dagger = x^{\underline{a}}, \quad (P_{\underline{a}})^\dagger = P_{\underline{a}}, \quad (S^i)^\dagger = S_i, \quad (\tilde{S}^i)^\dagger = \tilde{S}_i, \quad (3.9)$$

where, as before, $x^{\underline{a}}$ represent all the bosonic coordinates. We find that this process fixes the operator orderings for $\mathbb{P}^+, \mathbb{P}^x, \mathbb{P}^{\bar{x}}, \mathbb{Q}^{-i}, \mathbb{Q}^{-i}_-, \mathbb{K}^+, \mathbb{D}, \mathbb{J}^{+-}, \mathbb{J}^{x\bar{x}}, \mathbb{J}^{+x}, \mathbb{J}^{+\bar{x}}$ and \mathbb{J}^i_j . Next, demand that $\{\mathbb{Q}^{-i}, \mathbb{Q}^{-i}_-\}$ becomes proportional to δ^i_j . This turns out to fix the ordering of \mathbb{Q}^- . In this calculation, we made use of the following relation satisfied by the orbital part of the quantized $su(4)$ generators l^i_j :

$$l^i_j l^j_k = \frac{1}{4} \hat{l}^2 \delta^i_k + 2l^i_k, \quad \hat{l}^2 \equiv l^i_j l^j_i. \quad (3.10)$$

This identity, which appeared in [20], is discussed in the Appendix A and will be of importance again in the next section. The third step is to demand that $\{\mathbb{Q}^+, \mathbb{S}^-\}$ satisfy the correct algebra. This condition fixes the ordering of \mathbb{S}^- because the ordering ambiguity is proportional to the operator of the form $S/\sqrt{P_-}$. Up to this point all the fermionic generators have been fixed. The ordering of the remaining bosonic operators are then determined by requiring the proper closure of the $psu(2, 2|4)$ algebra. It turned out that only a part of the algebra was needed to fix the ambiguities but we have verified the remaining part as well for a good consistency check. Finally the forms of the operators at general τ can be computed by the unitary transformation $\mathcal{O}(\tau) = e^{\tau \mathbb{P}^-} \mathcal{O}(0) e^{-\tau \mathbb{P}^-}$, which actually terminates at order τ^2 .

We now display all the quantum Noether charges thus obtained, regarded as the generators of the four dimensional superconformal algebra. For convenience we use the following notations:

$$z \equiv e^\phi, \quad N_S = S^i S_i, \quad N_{\tilde{S}} = \tilde{S}^i \tilde{S}_i, \quad S \cdot \tilde{S} = S^i \tilde{S}_i. \quad (3.11)$$

z is the coordinate along the direction normal to the boundary of AdS , N_S and $N_{\tilde{S}}$ are the number operators for S^i and \tilde{S}^i respectively.

First, the translation generators are given by⁷

$$\mathbb{P}^x = iP_{\bar{x}}, \quad \mathbb{P}^{\bar{x}} = iP_x, \quad \mathbb{P}^+ = iP_-, \quad (3.12)$$

$$\mathbb{P}^- = \frac{i}{4P_-} \left[-4P_x P_{\bar{x}} + \partial_z^2 - \frac{1}{z} \partial_z + \frac{1}{z^2} (-3 - \hat{l}^2 + 4N_{\tilde{S}} - N_{\tilde{S}}^2 + 4l^m_k \tilde{S}^k \tilde{S}_m) \right]. \quad (3.13)$$

⁷Although we use the variable z instead of ϕ , it should be remembered that the hermiticity is still defined with respect to the ϕ variable. In particular, the hermitian momentum is $P_\phi = -i\partial_\phi = -i(1/z)\partial_z$.

The special conformal generators are

$$\mathbb{K}^x = -iz^2 P_{\bar{x}} + x \left(z\partial_z + ix^- P_- + ix P_x + \frac{1}{2}(N_S - N_{\tilde{S}} + 3) \right) + izS \cdot \tilde{S} - \tau \mathbb{J}^{-x}, \quad (3.14)$$

$$\mathbb{K}^{\bar{x}} = -iz^2 P_x + \bar{x} \left(z\partial_z + ix^- P_- + i\bar{x} P_{\bar{x}} + \frac{1}{2}(-N_S + N_{\tilde{S}} + 3) \right) + iz\tilde{S} \cdot S - \tau \mathbb{J}^{-\bar{x}}, \quad (3.15)$$

$$\mathbb{K}^+ = \frac{1}{i}(z^2 + x\bar{x})P_- + \tau(z\partial_z + ix P_x + i\bar{x} P_{\bar{x}} + 1 + \tau \mathbb{P}^-), \quad (3.16)$$

$$\begin{aligned} \mathbb{K}^- = & (x\bar{x} - z^2)\mathbb{P}^- + \bar{x}\mathbb{J}^{-x} + x\mathbb{J}^{-\bar{x}} + x^- z\partial_z + i(x^-)^2 P_- + 2x^- \\ & + \frac{i}{4P_-} \left[-2z\partial_z - 1 + 2N_{\tilde{S}} - N_{\tilde{S}}^2 - 2N_S + N_S^2 + 4(\tilde{S} \cdot S)(S \cdot \tilde{S}) \right. \\ & \left. + 4l_m^k (\tilde{S}^m \tilde{S}_k - S^m S_k) - 4z(P_{\bar{x}} \tilde{S} \cdot S + P_x S \cdot \tilde{S}) \right]. \end{aligned} \quad (3.17)$$

The dilatation operator takes the form

$$\mathbb{D} = -z\partial_z - (ix^- P_- + ix P_x + i\bar{x} P_{\bar{x}}) - \frac{3}{2} - \tau \mathbb{P}^-. \quad (3.18)$$

As for the Lorentz generators we get

$$\mathbb{J}^{+-} = -ix^- P_- + \frac{1}{2} + \tau \mathbb{P}^-, \quad (3.19)$$

$$\mathbb{J}^{x\bar{x}} = -i\bar{x} P_{\bar{x}} + ix P_x + \frac{1}{2}(N_S - N_{\tilde{S}}), \quad (3.20)$$

$$\mathbb{J}^{+x} = -ix P_- + i\tau P_{\bar{x}}, \quad (3.21)$$

$$\mathbb{J}^{+\bar{x}} = -i\bar{x} P_- + i\tau P_x, \quad (3.22)$$

$$\mathbb{J}^{-x} = -x\mathbb{P}^- + ix^- P_{\bar{x}} - \frac{P_{\bar{x}}}{2P_-}(N_S + N_{\tilde{S}} - 1) + \frac{i}{\sqrt{P_-}} S^k \mathbb{Q}_k^-, \quad (3.23)$$

$$\mathbb{J}^{-\bar{x}} = -\bar{x}\mathbb{P}^- + ix^- P_x + \frac{P_x}{2P_-}(N_S + N_{\tilde{S}} + 1) + \frac{i}{\sqrt{P_-}} \mathbb{Q}^{-k} S_k. \quad (3.24)$$

The $su(4)$ generators \mathbb{J}^i_j consist of the orbital part l^i_j and the spin part M^i_j :

$$\mathbb{J}^i_j = l^i_j + M^i_j. \quad (3.25)$$

l^i_j and M^i_j separately satisfy the same $su(4)$ algebra as \mathbb{J}^i_j . The explicit form of l^i_j is rather involved and is discussed in the Appendix A, together with its properties. On the other hand, the spin part is quite simple and is given by

$$M^i_j = S^i S_j - \frac{1}{4} \delta_j^i N_S + \tilde{S}^i \tilde{S}_j - \frac{1}{4} \delta_j^i N_{\tilde{S}}. \quad (3.26)$$

Finally the supertranslation and the superconformal generators are given by

$$\mathbb{Q}^{+i} = i\sqrt{P_-}S^i, \quad (3.27)$$

$$\mathbb{Q}_i^+ = -i\sqrt{P_-}S_i, \quad (3.28)$$

$$\mathbb{Q}^{-i} = \frac{i}{2\sqrt{P_-}} \left[2P_x S^i - \partial_z \tilde{S}^i + \frac{1}{z} \left(\tilde{S}^i (N_{\tilde{S}} - 1) - 2l^i_k \tilde{S}^k \right) \right], \quad (3.29)$$

$$\mathbb{Q}_i^- = \frac{-i}{2\sqrt{P_-}} \left[2P_{\bar{x}} S_i + \partial_z \tilde{S}_i + \frac{1}{z} \left(\tilde{S}_i (N_{\tilde{S}} - 3) - 2\tilde{S}_k l^k_i \right) \right], \quad (3.30)$$

$$\mathbb{S}^{+i} = -i\sqrt{P_-} \left(z\tilde{S}^i + i\bar{x}S^i \right) + i\tau\mathbb{Q}^{-i}, \quad (3.31)$$

$$\mathbb{S}_i^+ = i\sqrt{P_-} \left(z\tilde{S}_i - ixS_i \right) - i\tau\mathbb{Q}_i^-, \quad (3.32)$$

$$\mathbb{S}^{-i} = \frac{-i}{2\sqrt{P_-}} \left[2zP_{\bar{x}}\tilde{S}^i - 2\tilde{S}^i(S \cdot \tilde{S}) - S^i(z\partial_z + N_S + 1) + 2l^i_k S^k \right] + ix^-\mathbb{Q}^{+i} + ix\mathbb{Q}^{-i}, \quad (3.33)$$

$$\mathbb{S}_i^- = \frac{i}{2\sqrt{P_-}} \left[2zP_x\tilde{S}_i - 2\tilde{S}_i(\tilde{S} \cdot S) + S_i(z\partial_z - N_S + 5) + 2l^k_i S_k \right] - ix^-\mathbb{Q}_i^+ - i\bar{x}\mathbb{Q}_i^-. \quad (3.34)$$

Here we should remark that, although not all the generators were explicitly displayed, in [20] Metsaev ingeniously wrote down essentially the same form of generators, without systematic derivations.

Having derived the complete set of quantum generators for the $psu(2, 2|4)$ superconformal algebra, we are now ready to study the physical states which form unitary irreducible representations of this concrete system.

4 Solution of the superconformal primary states

Our strategy for studying the spectrum and other quantum properties of the superparticle in $AdS_5 \times S^5$ is to make maximal use of the representation theory of its symmetry algebra. The general theory of the representations of $psu(2, 2|4)$ algebra has been fairly well-developed[31, 32, 26, 33, 34] and the classification of all the unitary irreducible representations are known. They include special short and semi-short ‘‘BPS’’ representations, which have been realized in various parts of AdS/CFT correspondence. With such knowledge at hand, the problem we wish to solve is to find precisely which representations can be realized in the Hilbert space where the generators of $psu(2, 2|4)$ are realized in the specific form given in the previous section. In this section we will give a complete answer to this problem by constructing all possible superconformal primary states, including their explicit wavefunctions.

4.1 Dilatation (D) scheme and energy (E) scheme

To begin, it is important to discuss the two commonly used schemes of describing the representations of the conformal group $SO(4, 2)$ ⁸. They will be called the E-scheme and the D-scheme and are based on the following maximal subgroup decomposition:

$$\text{E-scheme} \quad SO(4, 2) \supset SO(2)_E \times SU(2)_L \times SU(2)_R, \quad (4.1)$$

$$\text{D-scheme} \quad SO(4, 2) \supset SO(1, 1)_D \times SL(2, C) \times \overline{SL(2, C)}. \quad (4.2)$$

Recall that in our convention, the embedding coordinates are labeled as $(X^{-1}, X^0, X^1, X^2, X^3, X^4)$ with the signature $(-, -, +, +, +, +)$. In the E-scheme, $SO(2)_E$ acts on the coordinates (X^{-1}, X^0) , while $SU(2)_L \times SU(2)_R \simeq SO(4)$ rotates (X^1, X^2, X^3, X^4) . The generator of $SO(2)_E$ is the hermitian AdS energy, to be denoted by \mathbb{E} . In terms of the generators in the light-cone basis, it is given by

$$\mathbb{E} = \frac{1}{i}T^{0,-1} = \frac{1}{2i}(\mathbb{P}^+ - \mathbb{P}^- - \mathbb{K}^+ + \mathbb{K}^-). \quad (4.3)$$

On the other hand, in the D-scheme $SO(1, 1)_D$ acts on (X^{-1}, X^4) and the Lorentz group $SL(2, C) \times \overline{SL(2, C)} \simeq SO(3, 1)$ acts on (X^0, X^1, X^2, X^3) . The generator of $SO(1, 1)$ is the dilatation operator \mathbb{D} . In our convention it is anti-hermitian.

Clearly, these two schemes are related by the exchange $X^0 \leftrightarrow X^4$, which is generated by the anti-hermitian boost operator

$$R \equiv T^{40} = \frac{1}{2}(\mathbb{P}^+ - \mathbb{P}^- + \mathbb{K}^+ - \mathbb{K}^-). \quad (4.4)$$

In fact, by using the basic commutation relations of $SO(4, 2)$, it is easy to see that \mathbb{D} is mapped to \mathbb{E} by a similarity transformation of the form

$$V\mathbb{D}V^{-1} = \mathbb{E}, \quad V = e^{i(\pi/2)R}. \quad (4.5)$$

Of course one can map any generator \mathcal{O} of $psu(2, 2|4)$ by this similarity transformation and we denote it by

$$V\mathcal{O}V^{-1} = \hat{\mathcal{O}}. \quad (4.6)$$

In this notation, $\mathbb{E} = \hat{\mathbb{D}}$. As it is a similarity transformation, this mapping preserves the structure of the superconformal algebra. However, it is important to note that it is a *non-unitary* transformation and hence it does not preserve the norm. Therefore, to obtain a unitary (hence normalizable) representation, one must choose an appropriate scheme.

⁸These two schemes were extensively discussed in [34].

As we expect to be able to reproduce the supergravity result, the proper scheme should be the E-scheme, with *real values for the AdS energy E* . This will be confirmed in the subsequent sections. Therefore, as for the $SO(4, 2)$ part, we will label the states by the eigenvalue of \mathbb{E} and those of the Cartan generators $\mathbb{J}_{L,R}^3$ of the $su(2)_{L,R}$ algebras. In terms of the light-cone basis generators, $\mathbb{J}_{L,R}^3$ are given by

$$\mathbb{J}_L^3 = \frac{1}{2}(\mathbb{H}_1 + \mathbb{H}_2), \quad \mathbb{J}_R^3 = \frac{1}{2}(\mathbb{H}_1 - \mathbb{H}_2), \quad (4.7)$$

where

$$\mathbb{H}_1 = \mathbb{J}^{x\bar{x}} = \hat{\mathbb{J}}^{x\bar{x}}, \quad \mathbb{H}_2 = \frac{i}{2}(\mathbb{P}^+ + \mathbb{P}^- + \mathbb{K}^+ + \mathbb{K}^-) = -\hat{\mathbb{J}}^{+-}. \quad (4.8)$$

The eigenvalues of the hermitian operators \mathbb{H}_1 and \mathbb{H}_2 will be denoted by h_1 and h_2 respectively.

The non-unitary nature of the similarity transformation above manifests itself most conspicuously in the following fact. Suppose $|E\rangle$ is a unit-normalized energy eigenstate with real non-zero eigenvalue E , *i.e.* $\mathbb{E}|E\rangle = E|E\rangle$. Then, $\mathbb{D}(V^{-1}|E\rangle) = V^{-1}(V\mathbb{D}V^{-1})|E\rangle = V^{-1}\mathbb{E}|E\rangle = E(V^{-1}|E\rangle)$. In other words, the state $V^{-1}|E\rangle$ is an eigenstate of an anti-hermitian generator \mathbb{D} with real eigenvalue E . As is well-known, this can only happen if $V^{-1}|E\rangle$ is of zero-norm. Indeed the norm of this state, which is $\langle E|V^{-2}|E\rangle$, vanishes since as one can easily show that $V^{-2}\mathbb{E} = -E V^{-2}$. In the context of AdS/CFT, this phenomenon is consistent with the fact that the gauge-invariant composite operators in the super-Yang-Mills theory carry real eigenvalues with respect to the anti-hermitian dilatation operator⁹. However, it is rather non-trivial that, group theoretically, what corresponds to a physical CFT operator with a definite dilatation charge is a zero-norm state on the AdS side, which is hard to interpret physically. It would be interesting to clarify this structure more deeply.

4.2 Superconformal primary conditions

We now formulate the problem of finding the superconformal primary states in the E-scheme more explicitly. In this scheme, the superconformal primary state $|\Psi\rangle$ is characterized by the following 16 conditions

$$\hat{\mathbb{S}}^{\pm i}|\Psi\rangle = 0, \quad \hat{\mathbb{S}}_i^{\pm}|\Psi\rangle = 0, \quad (4.9)$$

⁹We thank R. Janik for a discussion on this point.

where $\hat{\mathbb{S}}^{\pm i} = V\mathbb{S}^{\pm i}V^{-1}$ and $\hat{\mathbb{S}}_i^{\pm} = V\mathbb{S}_i^{\pm}V^{-1}$. From the form of R given in (4.4) and the basic commutation relations listed in section 2, we obtain

$$[R, \mathbb{S}^{\pm i}] = \mp \frac{i}{2} \mathbb{Q}^{\mp i}, \quad [R, \mathbb{S}_i^{\pm}] = \pm \frac{i}{2} \mathbb{Q}_i^{\mp}, \quad (4.10)$$

$$[R, \mathbb{Q}^{\pm i}] = \mp \frac{i}{2} \mathbb{S}^{\mp i}, \quad [R, \mathbb{Q}_i^{\pm}] = \pm \frac{i}{2} \mathbb{S}_i^{\mp}. \quad (4.11)$$

By repeatedly applying these commutation relations, we can easily obtain a formula such as $e^{i\theta R} \mathbb{S}^{+i} e^{-i\theta R} = \mathbb{S}^{+i} \cos(\theta/2) + \mathbb{Q}^{-i} \sin(\theta/2)$, etc. Setting $\theta = \pi/2$, we get the superconformal generators in the E-scheme as

$$\hat{\mathbb{S}}^{+i} = \frac{1}{\sqrt{2}} (\mathbb{S}^{+i} + \mathbb{Q}^{-i}), \quad \hat{\mathbb{S}}^{-i} = \frac{1}{\sqrt{2}} (\mathbb{S}^{-i} - \mathbb{Q}^{+i}), \quad (4.12)$$

$$\hat{\mathbb{S}}_i^{+} = \frac{1}{\sqrt{2}} (\mathbb{S}_i^{+} - \mathbb{Q}_i^{-}), \quad \hat{\mathbb{S}}_i^{-} = \frac{1}{\sqrt{2}} (\mathbb{S}_i^{-} + \mathbb{Q}_i^{+}). \quad (4.13)$$

The desendants of the irreducible representation are generated from the superconformal primary state by the repeated action of the E-scheme version of the supertranslation generators $\hat{\mathbb{Q}}$'s, which can be obtained in an entirely similar manner. They are given by

$$\hat{\mathbb{Q}}^{+i} = \frac{1}{\sqrt{2}} (\mathbb{Q}^{+i} + \mathbb{S}^{-i}), \quad \hat{\mathbb{Q}}^{-i} = \frac{1}{\sqrt{2}} (\mathbb{Q}^{-i} - \mathbb{S}^{+i}), \quad (4.14)$$

$$\hat{\mathbb{Q}}_i^{+} = \frac{1}{\sqrt{2}} (\mathbb{Q}_i^{+} - \mathbb{S}_i^{-}), \quad \hat{\mathbb{Q}}_i^{-} = \frac{1}{\sqrt{2}} (\mathbb{Q}_i^{-} + \mathbb{S}_i^{+}). \quad (4.15)$$

The $SO(4, 2)$ quantum numbers (E, h_1, h_2) carried by these $\hat{\mathbb{Q}}$ operators are

$$\hat{\mathbb{Q}}^{+i} : \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right), \quad \hat{\mathbb{Q}}^{-i} : \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right), \quad (4.16)$$

$$\hat{\mathbb{Q}}_i^{+} : \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right), \quad \hat{\mathbb{Q}}_i^{-} : \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad (4.17)$$

Because of the relation $\{\hat{\mathbb{S}}, \hat{\mathbb{S}}\} \sim \hat{\mathbb{K}}$, the conformal primary conditions $\hat{\mathbb{K}}^a |\Psi\rangle = 0$ are automatically satisfied by the superconformal primaries. In this sense, we need not impose them separately. However, as they will be useful in the subsequent analysis, we will briefly discuss their explicit forms. By applying the similarity transformation (4.6) to \mathbb{K}^a , we easily obtain the desired counterparts in the E-scheme:

$$\hat{\mathbb{K}}^x = \frac{1}{2} (\mathbb{K}^x - \mathbb{P}^x - i(\mathbb{J}^{+x} - \mathbb{J}^{-x})), \quad (4.18)$$

$$\hat{\mathbb{K}}^{\bar{x}} = \frac{1}{2} (\mathbb{K}^{\bar{x}} - \mathbb{P}^{\bar{x}} - i(\mathbb{J}^{+\bar{x}} - \mathbb{J}^{-\bar{x}})), \quad (4.19)$$

$$\hat{\mathbb{K}}^{+} = \frac{1}{2} (\mathbb{K}^{+} - \mathbb{P}^{-} - i(\mathbb{J}^{+-} - \mathbb{D})), \quad (4.20)$$

$$\hat{\mathbb{K}}^{-} = \frac{1}{2} (\mathbb{K}^{-} - \mathbb{P}^{+} - i(\mathbb{J}^{+-} + \mathbb{D})). \quad (4.21)$$

On (super)conformal primaries these generators vanish. It will be convenient to express this by the notation¹⁰ $\hat{\mathbb{K}}^a \approx 0$. Then, from $\hat{\mathbb{K}}^\pm \approx 0$ we can express \mathbb{P}^- and \mathbb{K}^- , which are among the most complicated generators, in terms of other simpler generators. Explicitly,

$$\mathbb{P}^- \approx \mathbb{K}^+ - i(\mathbb{J}^{+-} - \mathbb{D}), \quad (4.22)$$

$$\mathbb{K}^- \approx \mathbb{P}^+ + i(\mathbb{J}^{+-} + \mathbb{D}). \quad (4.23)$$

If we apply these relations to the AdS energy operator given in (4.3), we obtain a relation

$$\mathbb{E} \approx -i(\mathbb{P}^+ - \mathbb{K}^+ + i\mathbb{J}^{+-}). \quad (4.24)$$

This will be of use in the next subsection.

4.3 Allowed unitary highest weight representation for the $su(4)$ sector

When expressed in terms of the basic quantum variables of the superparticle, the superconformal primary conditions in the E-scheme formulated in the previous subsection are actually quite involved even at $\tau = 0$ and cannot be analyzed as they stand.

We now make two observations which will simplify the situation. The first observation is that the dependence on the S^5 coordinates and the derivatives with respect to them is only through the generators l^i_j of the orbital part of $su(4)$. This means that the Casimir operator \hat{l}^2 commutes with all the generators of the $psu(2, 2|4)$ and hence we can analyze the possible representations of this orbital part independently.

The second observation is that the aforementioned quadratic relation (3.10) satisfied by l^i_j is quite useful for our analysis. For the present purpose, we display it again in the following form:

$$\mathcal{L}^i_j \equiv l^i_k l^k_j - \frac{1}{4} \hat{l}^2 \delta^i_j - 2l^i_j = 0. \quad (4.25)$$

Existence of such product relations among the generators dictate that the structure of the representation module is correspondingly restricted.

To illustrate this in the simplest possible setting, consider the $su(2)$ algebra realized by the generators (J^3, J^\pm) made out of two sets of fermionic oscillators $(b_i, b^i)_{i=1,2}$ with the anticommutation relations $\{b_i, b^j\} = \delta^i_j$, $\{b^i, b^j\} = \{b_i, b_j\} = 0$ in the following way:

$$J^3 = \frac{1}{2}(b^1 b_1 - b^2 b_2), \quad J^+ = b^1 b_2, \quad J^- = b^2 b_1. \quad (4.26)$$

¹⁰This useful notation was introduced in [22].

As usual the highest weight module is constructed by acting J^- to a highest weight state $|j\rangle$ satisfying $J^+|j\rangle = 0$ and $J^3|j\rangle = j|j\rangle$. But because of the special form of the generators, there is an obvious product relation $J^-J^- = 0$. This clearly restricts the allowed representations to be utmost two dimensional. Indeed for this system two singlet representations with the highest weight states $|0\rangle$ and $b^1b^2|0\rangle$ and one doublet representation with the highest weight state $b^1|0\rangle$ are the only possible representations.

Let us now examine the consequences of the product relation (4.25). The fastest way is to use the explicit realization of l_j^i in terms of the Chevalley basis generators $(H_i, E_i^\pm)_{i=1,2,3}$, which satisfy the commutation relations

$$[H_i, H_j] = 0, \quad [E_i^+, E_j^-] = \delta_{ij} H_j \quad (4.27)$$

$$[H_i, E_j^\pm] = \pm K_{ji} E_j^\pm, \quad K_{ji} = \text{Cartan matrix}. \quad (4.28)$$

It reads

$$l_j^i = \begin{pmatrix} \frac{1}{4}(3H_1 + 2H_2 + H_3) & E_1^+ & [E_1^+, E_2^+] & [E_1^+, [E_2^+, E_3^+]] \\ E_1^- & \frac{1}{4}(-H_1 + 2H_2 + H_3) & E_2^+ & [E_2^+, E_3^+] \\ -[E_1^-, E_2^-] & E_2^- & -\frac{1}{4}(H_1 + 2H_2 - H_3) & E_3^+ \\ [E_1^-, [E_2^-, E_3^-]] & -[E_2^-, E_3^-] & E_3^- & -\frac{1}{4}(H_1 + 2H_2 + 3H_3) \end{pmatrix} \quad (4.29)$$

Since we are interested in what highest weight representations are allowed, we apply (4.25) onto a highest weight state $|\lambda_1, \lambda_2, \lambda_3\rangle$, where $\lambda_i (\geq 0)$ denote the Dynkin weights. Such a state is characterized by $E_i^+|\lambda_1, \lambda_2, \lambda_3\rangle = 0$ and $H_i|\lambda_1, \lambda_2, \lambda_3\rangle = \lambda_i|\lambda_1, \lambda_2, \lambda_3\rangle$.

Consider first the equation $\mathcal{L}_1^2|\lambda_1, \lambda_2, \lambda_3\rangle = 0$. Using the Chevalley basis expressions, the left hand side can be easily computed and the resultant equation is equivalent to

$$(\lambda_1 + 2\lambda_2 + \lambda_3 + 2)E_1^-|\lambda_1, \lambda_2, \lambda_3\rangle = 0. \quad (4.30)$$

Since the coefficient in front is non-vanishing, we get $E_1^-|\lambda_1, \lambda_2, \lambda_3\rangle = 0$. It means that the state must be a singlet of $su(2)$ along the direction 1 and hence we must have $\lambda_1 = 0$. In an entirely similar manner, the relation $\mathcal{L}_3^4|\lambda_1, \lambda_2, \lambda_3\rangle = 0$ dictates $\lambda_3 = 0$. One can then check that the rest of the relations are automatically satisfied and do not lead to any further restrictions. Summarizing, the allowed highest weight states for the orbital part of $su(4)$ are of the form

$$|0, l, 0\rangle, \quad l = 0, 1, 2, \dots \quad (4.31)$$

From (4.29) we can easily work out the action of the generators l^i_j on this state. In particular, the following relations will be useful later:

$$\begin{aligned} l^i_j|0, l, 0\rangle &= 0 \quad \text{for } i < j, \quad l^2_1|0, l, 0\rangle = l^4_3|0, l, 0\rangle = 0, \\ l^1_1|0, l, 0\rangle &= l^2_2|0, l, 0\rangle = -l^3_3|0, l, 0\rangle = -l^4_4|0, l, 0\rangle = \frac{1}{2}l|0, l, 0\rangle, \\ \hat{l}^2|0, l, 0\rangle &= l(l+4)|0, l, 0\rangle. \end{aligned} \quad (4.32)$$

It should be noted that the analysis above can be regarded as an efficient algebraic means for performing the harmonic analysis on S^5 .

Next, we wish to apply a similar analysis of the representation of $su(4)$ to the entire Hilbert space, including the spin part. Here we encounter a difficulty: The total generators $\mathbb{J}^i_j = l^i_j + M^i_j$ do not satisfy any simple product relation such as (4.25). The reason for this is simple. The spin part of the Hilbert space is generated by the action of the eight creation operators S^i, \tilde{S}^i and hence consists of 2^8 states. These states fall into a large number of different highest weight representations. Therefore, tensored with the orbital part, the representations in the entire Hilbert space are not so severely restricted as in the orbital case.

Fortunately, however, product relations similar to (4.25) do exist *on superconformal primary states*. Specifically, consider the following linear combination of superconformal primary conditions:

$$\sqrt{P_-} \left(z\tilde{S}_j\hat{\mathbb{S}}^{+i} - z\tilde{S}^i\hat{\mathbb{S}}^{+}_j - S_j\hat{\mathbb{S}}^{-i} + S^i\hat{\mathbb{S}}^{-}_j \right) \approx 0. \quad (4.33)$$

Substituting the explicit form of the $\hat{\mathbb{S}}$ generators and making use of the formula (4.24), we obtain, after some computation, a useful relation

$$\begin{aligned} &\tilde{S}^i\tilde{S}_j(1 - N_{\tilde{S}}) + S^iS_j(1 - N_S) - \tilde{S}^i(S \cdot \tilde{S})S_j - S^i(\tilde{S} \cdot S)\tilde{S}_j \\ &+ \frac{1}{2}\delta^i_j(N_S + N_{\tilde{S}}) + l^i_k(S^kS_j + \tilde{S}^k\tilde{S}_j) + l^k_j(S^iS_k + \tilde{S}^i\tilde{S}_k) - 2l^i_j - \mathbb{E}\delta^i_j \approx 0. \end{aligned} \quad (4.34)$$

This in turn can be used to compute $\hat{\mathbb{J}}^i_k\hat{\mathbb{J}}^k_j$. After some computation the result can be put into the form

$$\mathcal{J}^i_j \equiv \hat{\mathbb{J}}^i_k\hat{\mathbb{J}}^k_j - \frac{1}{4}\hat{\mathbb{J}}^2\delta^i_j - \left(4 - \frac{N}{2}\right)\hat{\mathbb{J}}^i_j \approx 0, \quad (4.35)$$

where

$$N \equiv N_S + N_{\tilde{S}}, \quad (N = 0, 1, \dots, 8), \quad (4.36)$$

$$\hat{\mathbb{J}}^2 \equiv \mathbb{J}^i_k\mathbb{J}^k_i \approx 4\mathbb{E} + \hat{l}^2 - \frac{1}{4}(N - 4)^2 + 4. \quad (4.37)$$

Since the relation (4.35) has the form quite similar to (4.25), we can repeat the analysis for each value of N . In fact as each $\hat{\mathbb{S}}$ operator has a definite N -number, it is not possible to form a superconformal primary state by making a linear combination of non-superconformal primary states with different N -numbers. This means that the basic irreducible superconformal primary state must carry a definite value of N .

In this analysis, in order to narrow down the allowed representations, it is useful to impose the unitarity requirement at the same time. From the hermiticity property of \mathbb{Q} 's and \mathbb{S} 's, we easily find the hermiticity of their counterparts in the E-scheme, namely $\hat{\mathbb{Q}}$'s and $\hat{\mathbb{S}}$'s, as follows:

$$(\hat{\mathbb{Q}}^{+i})^\dagger = \hat{\mathbb{S}}_i^-, \quad (\hat{\mathbb{Q}}_i^+)^\dagger = -\hat{\mathbb{S}}^{-i} \quad (4.38)$$

$$(\hat{\mathbb{Q}}^{-i})^\dagger = -\hat{\mathbb{S}}_i^+, \quad (\hat{\mathbb{Q}}_i^-)^\dagger = \hat{\mathbb{S}}^{+i}. \quad (4.39)$$

Now let $|\Psi\rangle$ be a superconformal primary state, which is annihilated by $\hat{\mathbb{S}}$'s. Then for a unitary representation, we get a so-called unitarity bound by

$$\langle \Psi | \{ \hat{\mathbb{S}}_i^-, \hat{\mathbb{Q}}^{+i} \} | \Psi \rangle = |\hat{\mathbb{Q}}^{+i} | \Psi \rangle|^2 \geq 0, \quad (4.40)$$

and further bounds using other pairs. One can easily evaluate the anticommutator such as $\{ \hat{\mathbb{S}}_i^-, \hat{\mathbb{Q}}^{+i} \}$ by transforming the known result for $\{ \mathbb{S}_i^-, \mathbb{Q}^{+i} \}$. If the $su(4)$ part of $|\Psi\rangle$ is taken to be $|\lambda_1, \lambda_2, \lambda_3\rangle$, we can evaluate the left-hand-side of (4.40) explicitly. In this way, one can obtain a useful bound such as $E \geq \lambda_1 + \lambda_2 + \lambda_3$, which can be used to eliminate a number of possible representations during the analysis of (4.35). Since the detail is somewhat involved we relegate it to the Appendix C.

The outcome of this analysis is that the allowed highest weight states for the $su(4)$ sector can only be of the following three types:

$$(i) \quad |\Omega_l\rangle = S^1 \tilde{S}^1 S^2 \tilde{S}^2 |0\rangle \otimes |0, l, 0\rangle, \quad l = 0, 1, 2, \dots, \quad (4.41)$$

$$(ii) \quad |\text{vac}\rangle = |0\rangle \otimes |0, 0, 0\rangle, \quad (4.42)$$

$$(iii) \quad |\text{fvac}\rangle = S^1 \tilde{S}^1 S^2 \tilde{S}^2 S^3 \tilde{S}^3 S^4 \tilde{S}^4 |0\rangle \otimes |0, 0, 0\rangle. \quad (4.43)$$

The first factor of the tensor product is the spin part and the second factor is the orbital part. The symbols “vac” and “fvac” signify the vacuum and the filled-vacuum nature of the spin part. In the next subsection, we will show that only the states of type (i) will lead to the proper normalizable superconformal primary states.

4.4 Solutions and properties of the superconformal primary states

4.4.1 Solutions of the superconformal primaries at $\tau = 0$

We are now ready to solve the superconformal primary conditions explicitly. In this subsection, we will concentrate on the solutions at $\tau = 0$.

First consider the states built upon $|\Omega_l\rangle$ given in (4.41). We will write a state of this type as

$$|\Psi_l\rangle = \Phi_l(z, P_-, P_x, P_{\bar{x}})|\Omega_l\rangle. \quad (4.44)$$

On such a state, the supercharge operators simplify substantially. Since half of the fermionic oscillators are excited in $|\Omega_l\rangle$, below we will split the $su(4)$ index i as $i = (\alpha, \hat{\alpha})$, where $\alpha = 1, 2$ and $\hat{\alpha} = 3, 4$. Then, the fermionic oscillators act on $|\Omega_l\rangle$ as

$$S_{\hat{\alpha}}|\Omega_l\rangle = \tilde{S}_{\hat{\alpha}}|\Omega_l\rangle = S^{\alpha}|\Omega_l\rangle = \tilde{S}^{\alpha}|\Omega_l\rangle = 0, \quad (4.45)$$

$$N_S|\Omega_l\rangle = N_{\tilde{S}}|\Omega_l\rangle = 2|\Omega_l\rangle, \quad (4.46)$$

$$\tilde{S} \cdot S|\Omega_l\rangle = S \cdot \tilde{S}|\Omega_l\rangle = 0. \quad (4.47)$$

As for the structures involving l^i_k , using (4.32) we get

$$l^{\hat{\alpha}}_k S^k|\Omega_l\rangle = -\frac{l}{2} S^{\hat{\alpha}}|\Omega_l\rangle, \quad l^{\alpha}_k S^k|\Omega_l\rangle = 0, \quad (4.48)$$

$$l^{\hat{\alpha}}_k \tilde{S}^k|\Omega_l\rangle = -\frac{l}{2} \tilde{S}^{\hat{\alpha}}|\Omega_l\rangle, \quad l^{\alpha}_k \tilde{S}^k|\Omega_l\rangle = 0, \quad (4.49)$$

$$S_k l^k_{\alpha}|\Omega_l\rangle = \frac{l}{2} S_{\alpha}|\Omega_l\rangle, \quad S_k l^k_{\hat{\alpha}}|\Omega_l\rangle = 0, \quad (4.50)$$

$$\tilde{S}_k l^k_{\alpha}|\Omega_l\rangle = \frac{l}{2} \tilde{S}_{\alpha}|\Omega_l\rangle, \quad \tilde{S}_k l^k_{\hat{\alpha}}|\Omega_l\rangle = 0. \quad (4.51)$$

Applying these results to the supercharges \mathbb{Q} 's and \mathbb{S} 's, we find that they simplify considerably and effectively reduce to the following forms on $|\Psi_l\rangle$:

$$\mathbb{Q}^{\pm\alpha} = 0, \quad \mathbb{Q}_{\hat{\alpha}}^{\pm} = 0, \quad (4.52)$$

$$\mathbb{Q}^{+\hat{\alpha}} = i\sqrt{P_-} S^{\hat{\alpha}}, \quad \mathbb{Q}_{\alpha}^{+} = -i\sqrt{P_-} \tilde{S}_{\alpha}, \quad (4.53)$$

$$\mathbb{Q}^{-\hat{\alpha}} = \frac{i}{2\sqrt{P_-}} \left(2P_x S^{\hat{\alpha}} - \left(\partial_z - \frac{l+1}{z} \right) \tilde{S}^{\hat{\alpha}} \right), \quad (4.54)$$

$$\mathbb{Q}_{\alpha}^{-} = \frac{-i}{2\sqrt{P_-}} \left(2P_{\bar{x}} S_{\alpha} + \left(\partial_z - \frac{l+1}{z} \right) \tilde{S}_{\alpha} \right), \quad (4.55)$$

$$\mathbb{S}^{\pm\alpha} = 0, \quad \mathbb{S}_{\hat{\alpha}}^{\pm} = 0, \quad (4.56)$$

$$\mathbb{S}^{+\hat{\alpha}} = -i\sqrt{P_-} \left(z\tilde{S}^{\hat{\alpha}} - \frac{\partial}{\partial P_{\bar{x}}} S^{\hat{\alpha}} \right), \quad (4.57)$$

$$\mathbb{S}_{\alpha}^{+} = i\sqrt{P_-} \left(z\tilde{S}_{\alpha} + \frac{\partial}{\partial P_x} S_{\alpha} \right), \quad (4.58)$$

$$\mathbb{S}^{-\hat{\alpha}} = \frac{-i}{2\sqrt{P_-}} \left[2zP_{\bar{x}}\tilde{S}^{\hat{\alpha}} - (z\partial_z + l + 3)S^{\hat{\alpha}} \right] - \frac{\partial}{\partial P_-} \mathbb{Q}^{+\hat{\alpha}} - \frac{\partial}{\partial P_x} \mathbb{Q}^{-\hat{\alpha}}, \quad (4.59)$$

$$\mathbb{S}_{\alpha}^{-} = \frac{i}{2\sqrt{P_-}} \left[2zP_x\tilde{S}_{\alpha} + (z\partial_z + l + 3)S_{\alpha} \right] + \frac{\partial}{\partial P_-} \mathbb{Q}_{\alpha}^{+} + \frac{\partial}{\partial P_{\bar{x}}} \mathbb{Q}_{\alpha}^{-}. \quad (4.60)$$

Note that, combining (4.52) and (4.56), the following half of the superconformal primary conditions in the E-scheme are automatically satisfied:

$$\hat{\mathbb{S}}^{\pm\alpha}|\Psi_l\rangle = 0, \quad \hat{\mathbb{S}}_{\hat{\alpha}}^{\pm}|\Psi_l\rangle = 0. \quad (4.61)$$

Similarly, we see that the half of the supercharges in the E-scheme annihilate $|\Psi_l\rangle$:

$$\hat{\mathbb{Q}}^{\pm\alpha}|\Psi_l\rangle = 0, \quad \hat{\mathbb{Q}}_{\hat{\alpha}}^{\pm}|\Psi_l\rangle = 0. \quad (4.62)$$

This means that *all the highest weight representations of this type are half BPS*.

We now impose the remaining superconformal primary conditions one by one to determine the form of Φ_l .

First, consider the condition

$$\begin{aligned} 0 &= \sqrt{2}\hat{\mathbb{S}}^{+\hat{\alpha}}|\Psi_l\rangle = (\mathbb{S}^{+\hat{\alpha}} + \mathbb{Q}^{-\hat{\alpha}})|\Psi_l\rangle \\ &= \frac{i}{2\sqrt{P_-}} \left[2 \left(P_x + P_- \frac{\partial}{\partial P_{\bar{x}}} \right) S^{\hat{\alpha}} - \left(\partial_z - \frac{l+1}{z} + 2P_-z \right) \tilde{S}^{\hat{\alpha}} \right] |\Psi_l\rangle. \end{aligned} \quad (4.63)$$

From the coefficient of $S^{\hat{\alpha}}$ and $\tilde{S}^{\hat{\alpha}}$, we get two first order differential equations:

$$\left(P_x + P_- \frac{\partial}{\partial P_{\bar{x}}} \right) \Phi_l = 0, \quad (4.64)$$

$$\left(\partial_z - \frac{l+1}{z} + 2P_-z \right) \Phi_l = 0. \quad (4.65)$$

The first equation determines the $P_{\bar{x}}$ dependence and gives

$$\Phi_l = f_1(z, P_x, P_-) \exp \left(-\frac{P_x P_{\bar{x}}}{P_-} \right). \quad (4.66)$$

The second equation on the other hand determines the dependence on z and gives

$$\Phi_l = f_2(P_-, P_x, P_{\bar{x}}) \exp(-z^2 P_-) z^{l+1}. \quad (4.67)$$

Combining, we get

$$\Phi_l = f(P_-)\psi, \quad (4.68)$$

$$\psi = \exp\left(-\frac{P_x P_{\bar{x}}}{P_-} - z^2 P_-\right) z^{l+1}. \quad (4.69)$$

Next consider the following condition

$$0 = \sqrt{2} \hat{S}^{-\hat{\alpha}} |\Psi_l\rangle = (\mathbb{S}^{-\hat{\alpha}} - \mathbb{Q}^{+\hat{\alpha}}) |\Psi_l\rangle. \quad (4.70)$$

After some calculations one obtains

$$(\mathbb{S}^{-\hat{\alpha}} - \mathbb{Q}^{+\hat{\alpha}}) |\Psi_l\rangle = -\frac{i}{\sqrt{P_-}} \left[(1+z^2)P_- - \left(l + \frac{1}{2}\right) - \frac{P_x P_{\bar{x}}}{P_-} + P_- \frac{\partial}{\partial P_-} \right] S^{\hat{\alpha}} |\Psi_l\rangle = 0. \quad (4.71)$$

Plugging in the form of $\Phi = f(P_-)\psi$ above, we get the equation for $f(P_-)$ of the form

$$\frac{\partial}{\partial P_-} f = \left(\frac{l + \frac{1}{2}}{P_-} - 1 \right) f. \quad (4.72)$$

This is readily solved to give

$$f = C_l e^{-P_-} P_-^{l+(1/2)}, \quad (4.73)$$

where C_l is a constant.

Finally, one can easily check that the remaining conditions $0 = \sqrt{2} \hat{S}_\alpha^+ |\Psi_l\rangle = (\mathbb{S}_\alpha^+ - \mathbb{Q}_\alpha^-) |\Psi_l\rangle$ and $0 = \sqrt{2} \hat{S}_\alpha^- |\Psi_l\rangle = (\mathbb{S}_\alpha^- + \mathbb{Q}_\alpha^+) |\Psi_l\rangle$ are satisfied automatically.

Summarizing, we have found that upon $|\Omega_l\rangle$ a unique superconformal primary state exists for each l , which takes the form¹¹

$$\begin{aligned} |\Psi_l\rangle &= C_l \exp\left(-\frac{P_x P_{\bar{x}}}{P_-} - (z^2 + 1)P_-\right) z^{l+1} P_-^{l+(1/2)} \\ &\times S^1 \tilde{S}^1 S^2 \tilde{S}^2 |0\rangle \otimes |0, l, 0\rangle, \quad l = 0, 1, 2, \dots \end{aligned} \quad (4.74)$$

The quantum numbers of this state are read off by acting $\mathbb{E}, \mathbb{J}_{L,R}^3$ and $\hat{\mathbb{J}}^2$ on this state. We obtain

$$\mathbb{E} |\Psi_l\rangle = E_l |\Psi_l\rangle, \quad E_l = l + 2, \quad (4.75)$$

$$\mathbb{J}_{L,R}^3 |\Psi_l\rangle = 0, \quad (4.76)$$

$$\hat{\mathbb{J}}^2 |\Psi_l\rangle = (l+2)(l+6) |\Psi_l\rangle. \quad (4.77)$$

¹¹For some special states belonging to $l = 0$ multiplet, Metsaev obtained the bosonic part of the wave function in [22].

The descendants belonging to the highest weight representation are produced by operating the 8 supercharges $\hat{\mathbb{Q}}^{\pm 3,4}, \hat{\mathbb{Q}}_{1,2}^{\pm}$ (and the momentum operators $\hat{\mathbb{P}}$'s) on $|\Psi_l\rangle$. Mapping the basic commutation relations $[\mathbb{D}, \mathbb{Q}] = \frac{1}{2}\mathbb{Q}$ to the E-scheme, we obtain $[\mathbb{E}, \hat{\mathbb{Q}}] = \frac{1}{2}\hat{\mathbb{Q}}$. So each time we act by a $\hat{\mathbb{Q}}$, the AdS energy is raised by $\frac{1}{2}$ unit. For example, one of the first excited states is of the form $\hat{\mathbb{Q}}^{+3}|\Psi_l\rangle = \sqrt{P_-/2S^3}|\Psi_l\rangle$, which carries the energy $l + (3/2)$. From the form of $|\Psi_l\rangle$, the dimension of the representation (up to the action of $\hat{\mathbb{P}}$'s) is readily seen to be given by

$$2^8 \times \dim [0, l, 0] = \frac{64}{3}(l+1)(l+2)^2(l+3). \quad (4.78)$$

It is precisely that of the $\frac{1}{2}$ BPS superconformal multiplets of 1-particle states realized in type IIB supergravity[25, 26]. In the AdS/CFT context, these states correspond to the single trace operator $\text{Tr}(\phi^{\{I_1}\phi^{I_2}\dots\phi^{I_{l+2}\}})$ and its descendants in $N = 4$ Super-Yang-Mills theory.

In supergravity, one often speaks of the “mass” formula, which expresses the eigenvalue of the D’Alembertian for the AdS space. Although it is not a genuine invariant of the entire $psu(2, 2|4)$ algebra, we expect it to be related to the value of the quadratic Casimir operator $\frac{1}{2}T^{AB}T_{AB}$ of the $so(4, 2)$ subalgebra. Indeed, on $|\Psi_l\rangle$ we find the well-known formula

$$\frac{1}{2}T^{AB}T_{AB}|\Psi_l\rangle = E_l(E_l - 4)|\Psi_l\rangle. \quad (4.79)$$

The states $|\Psi_l\rangle$ are normalizable in the standard quantum mechanical sense¹² namely with respect to the integration measure which respects the hermiticity of the basic variables. Explicitly, the squared norm of $|\Psi_l\rangle$ is given by

$$\int_0^\infty \frac{dz}{z} \int_0^\infty dP_- \int_{-\infty}^\infty dP_1 \int_{-\infty}^\infty dP_2 \langle \Psi_l | \Psi_l \rangle = |C_l|^2 \frac{(l+1)(l!)^2}{2^{2l+4}} \pi. \quad (4.80)$$

Let us make two remarks. First in our scheme the measure for z variable should be taken to be $dz/z = d\phi$, since ϕ and its conjugate P_ϕ were regarded as the basic hermitian variables. Second, the range of P_- should be taken to be the semi-infinite interval $[0, \infty]$. The reason is that $|\Psi_l\rangle$ vanishes at both ends of this interval and this insures the hermiticity of x^- and P_- .

Having constructed the series of states $|\Psi_l\rangle$ upon $|\Omega_l\rangle$, let us now consider the superconformal primary state built upon $|\text{vac}\rangle$ given in (4.42). On $|\text{vac}\rangle$, S_i , \tilde{S}_i and all the

¹²Note that we are dealing with a quantum wave function of a particle and not with a supergravity field. So the measure should be taken to be the one appropriate for quantum mechanical interpretation.

orbital $su(4)$ generators l_j^i vanish. This immediately leads to $Q_i^+ = Q_i^- = S_i^+ = S_i^- = 0$. The remaining supercharges effectively take the form

$$Q^{+i} = i\sqrt{P_-}S^i, \quad (4.81)$$

$$Q^{-i} = \frac{i}{2\sqrt{P_-}} \left(2P_x S^i - \left(\partial_z + \frac{1}{z} \right) \tilde{S}^i \right), \quad (4.82)$$

$$S^{+i} = -i\sqrt{P_-} \left(z\tilde{S}^i - \frac{\partial}{\partial P_{\bar{x}}} S^i \right), \quad (4.83)$$

$$S^{-i} = \frac{-i}{2\sqrt{P_-}} \left[2zP_{\bar{x}}\tilde{S}^i - (z\partial_z + 1)S^i \right] - \frac{\partial}{\partial P_-} Q^{+i} - \frac{\partial}{\partial P_x} Q^{-i}. \quad (4.84)$$

Note that the form of Q^{+i} and S^{+i} are the same as (4.53) and (4.57) for $i = \hat{\alpha}$, while Q^{-i} and S^{-i} coincide with (4.54) and (4.59) for $i = \hat{\alpha}$ if we set $l = -2$. With this in mind, going through the analysis parallel to the previous case, we easily find that the primary state on $|\text{vac}\rangle$ is of the form

$$|\Psi_{\text{vac}}\rangle = \Phi_{l=-2}|\text{vac}\rangle \propto \exp \left(-\frac{P_x P_{\bar{x}}}{P_-} - (z^2 + 1)P_- \right) z^{-1} P_-^{-3/2} |\text{vac}\rangle. \quad (4.85)$$

As indicated, the part other than $|\text{vac}\rangle$ is identical to $|\Psi_l\rangle$ for $l = -2$ and hence this state has $E = 0$. It is however no longer normalizable: The integral over z for $\langle \Psi_{\text{vac}} | \Psi_{\text{vac}} \rangle$ behaves like $\sim \int dz/z^3$ near $z = 0$ and is divergent.

The analysis for the state built upon $|\text{fvac}\rangle$ given in (4.43) is very similar. The wave function is identical to (4.85) above except $|\text{vac}\rangle$ is replaced by $|\text{fvac}\rangle$. Such a state is also not normalizable.

4.4.2 Complete solution at arbitrary τ

Although the spectrum and the quantum numbers can be read off from $|\Psi_l\rangle$ at $\tau = 0$, it is of interest to compute the full wave function $|\Psi_l(\tau)\rangle$ at arbitrary τ in order to see the profile of the wave function in the AdS space and gain physical understanding. As already explained previously, it is obtained from the solution $|\Psi_l\rangle$ at $\tau = 0$ by the unitary transformation

$$|\Psi_l(\tau)\rangle = e^{\tau \mathbb{P}^-} |\Psi_l\rangle. \quad (4.86)$$

Upon $|\Psi_l\rangle$ the operator \mathbb{P}^- simplifies to

$$\begin{aligned} \mathbb{P}^- &= \frac{i}{4P_-} \left(D_z^{(l)} - 4P_x P_{\bar{x}} \right), \\ D_z^{(l)} &\equiv \partial_z^2 - \frac{1}{z} \partial_z - \frac{l^2 - 1}{z^2}. \end{aligned} \quad (4.87)$$

Nevertheless, the direct evaluation of the unitary transformation above is still difficult. A standard trick is to convert it to a Schrödinger equation by differentiating it with respect to τ . We obtain $\partial_\tau |\Psi_l(\tau)\rangle = \mathbb{P}^- |\Psi_l(\tau)\rangle$, which can be rewritten as

$$4 \left(\frac{1}{i} P_- \partial_\tau + P_x P_{\bar{x}} \right) |\Psi_l(\tau)\rangle = D_z^{(l)} |\Psi_l(\tau)\rangle. \quad (4.88)$$

Now one can easily check that the eigenfunction of the differential operator $D_z^{(l)}$ is given by

$$D_z^{(l)} f_l(\beta, z) = -\beta^2 f_l(\beta, z), \quad (4.89)$$

$$f_l(\beta, z) = z J_l(\beta z), \quad (4.90)$$

where $J_l(x)$ is the standard Bessel function of order l . Moreover, a very useful integration formula involving $J_l(\beta z)$ exists. It reads

$$\int_0^\infty d\beta e^{-\beta^2/4P_-} \beta^{l+1} J_l(\beta z) = z^l 2^{l+1} P_-^{l+1} e^{-z^2 P_-}. \quad (4.91)$$

It allows us to express the solution $|\Psi_l\rangle$ given in (4.74) as the following integral

$$|\Psi_l\rangle = \int_0^\infty d\beta \psi_0(\beta) f_l(\beta, z) |\Omega_l\rangle, \quad (4.92)$$

$$\psi_0(\beta) = C_l 2^{-(l+1)} P_-^{-1/2} \beta^{l+1} \exp\left(-\frac{\beta^2}{4P_-}\right) \exp\left(-\frac{P_x P_{\bar{x}}}{P_-} - P_-\right). \quad (4.93)$$

This suggests that we should seek the solution of the Schrödinger equation in the form

$$|\Psi_l(\tau)\rangle = \int_0^\infty d\beta \psi(\beta, \tau) f_l(\beta, z) |\Omega_l\rangle, \quad (4.94)$$

where the function $\psi(\beta, \tau)$ should satisfy the initial condition $\psi(\beta, \tau = 0) = \psi_0(\beta)$. Putting (4.94) into (4.88), one obtains a simple first order differential equation with respect to τ for $\psi(\beta, \tau)$, which can be readily solved. The solution obeying the initial condition is

$$\psi(\beta, \tau) = \psi_0(\beta) \chi(\beta, \tau), \quad (4.95)$$

$$\chi(\beta, \tau) = \exp\left(\frac{-i\tau\beta^2}{4P_-}\right) \exp\left(-i\frac{P_x P_{\bar{x}}}{P_-} \tau\right). \quad (4.96)$$

Since $\chi(\beta, \tau)$ is a Gaussian in β , we can perform the integral (4.94) by using the formula (4.91) again. In this way, we obtain $|\Psi_l(\tau)\rangle$ in a closed form. Reinstating the original relation $\tau = x^+$, it reads

$$\begin{aligned} |\Psi_l(z, P_x, P_{\bar{x}}, P_-, x^+)\rangle &= C_l \left(\frac{z}{1 + ix^+} \right)^{l+1} P_-^{l+(1/2)} \\ &\times \exp\left(-\frac{P_x P_{\bar{x}}}{P_-} (1 + ix^+) - P_- - \frac{z^2 P_-}{1 + ix^+}\right) |\Omega_l\rangle. \end{aligned} \quad (4.97)$$

To get a feel for this wave function, let us perform the Fourier transform to go to the full coordinate representation. The transforms with respect to P_x and $P_{\bar{x}}$ are standard and yield

$$|\Psi_l(z, x, \bar{x}, P_-, x^+)\rangle = C_l z^{l+1} (1 + ix^+)^{-(l+2)} P_-^{l+(3/2)} e^{-\alpha P_-} |\Omega_l\rangle, \quad (4.98)$$

$$\alpha \equiv \frac{x\bar{x} + z^2 + 1 + ix^+}{1 + ix^+}. \quad (4.99)$$

As for the transform with respect to P_- , we employ the general formulas for the semi-infinite interval, namely

$$\tilde{f}(x) = \int_0^\infty \frac{dp}{\sqrt{2\pi}} e^{ipx} f(p), \quad f(p) = \int_{-\infty}^\infty \frac{dx}{\sqrt{2\pi}} e^{-ipx} \tilde{f}(x). \quad (4.100)$$

In this way the full coordinate representation is obtained as

$$\begin{aligned} |\Psi_l(z, x, \bar{x}, x^-, x^+)\rangle &= C_l z^{l+1} (1 + ix^+)^{-(l+2)} \int_0^\infty \frac{dP_-}{\sqrt{2\pi}} e^{-\alpha P_-} P_-^{l+(3/2)} e^{ix^- P_-} |\Omega_l\rangle \\ &= \frac{2\sqrt{2}\pi^2 (2l+3)!!}{l! \sqrt{l+1}} z^{l+1} (1 + ix^+)^{1/2} \\ &\quad \times (x\bar{x} + x^+ x^- + z^2 + 1 + i(x^+ - x^-))^{-(l+(5/2))} |\Omega_l\rangle. \end{aligned} \quad (4.101)$$

The probability distribution takes the form

$$|\Psi_l(z, x, \bar{x}, x^-, x^+)\rangle^2 \propto \frac{z^{2l+2} (1 + (x^+)^2)^{1/2}}{[(x\bar{x} + x^+ x^- + z^2 + 1)^2 + (x^+ - x^-)^2]^{l+(5/2)}}. \quad (4.102)$$

The rough profile of this distribution is as follows. First, it vanishes when any one of the variables becomes large in its magnitude and this occurs more rapidly for higher l . It also vanishes, like z^{2l+2} , near the boundary $z = 0$. On the other hand, it tends to a constant when x 's become small. The fall off of $|\Psi_l\rangle$ as $\sim z^{l+1}$ near the boundary can be understood more physically from the Schrödinger equation (4.88). The structure of the Hamiltonian $H^{l.c.} = i\mathbb{P}^-$ is rather similar to that of a system in a centrifugal potential depending on the angular momentum l , such as the hydrogen atom. Near $z = 0$, the requirement of the absence of singularity dictates the wave function to be of the form z^α , where α satisfies $-\alpha^2 + 2\alpha + l^2 - 1 = 0$. Thus for the normalizable solution we must have $\alpha = l + 1$.

5 Discussions

In this work, we have succeeded in quantizing a superparticle in the $AdS_5 \times S^5$ background with RR flux exactly and obtained the complete spectrum of one-particle states. It is

gratifying that the result precisely agreed with that of supergravity, although the method is totally different.

There are two major directions for future research. One is the understanding of the GKP-W relation from the first-quantized viewpoint. In this regard, we note an important apparent difference between the supergravity analysis and our analysis. The equations of motion for the supergravity fields are second order in the derivative with respect to the AdS coordinates and hence one obtains two solutions. One is normalizable (under a norm appropriate in field theory) and corresponds to the propagating particle mode. The other is non-normalizable and is thought to play the role of the source for the gauge-invariant super-Yang-Mills operator placed on the boundary of AdS . In contrast, in our approach, the superconformal primary condition is an equation linear in the derivative and hence we obtained a unique solution as the highest weight state of the unitary representation. It is normalizable as a quantum mechanical wave function. It is not clear to us at the moment whether we should look for the missing “non-normalizable” states. One reason is that the first quantized approach inherently deals with a physical particle and hence non-particle mode may not be described. Another reason is that if we can construct the vertex operators anchored at points on the boundary which carry the quantum numbers of the corresponding particle modes, it should suffice to compute their correlation functions to see if they correspond to those in the super-Yang-Mills theory. If this is successful, one will not need the non-normalizable states, at least explicitly. In any case, construction of the appropriate vertex operators will be a major goal in this direction.

Another important direction, of course, is the extension of our method to the superstring case. The first task is the construction of the appropriately normal-ordered quantum superconformal generators. Once they are obtained, one can start solving the superconformal primary conditions. Due to the presence of the non-zero modes, the $su(4)$ part of the wave function will not be unique in contrast to the particle case and this will lead to many solutions for the superconformal primaries. Nevertheless we may hope that, perhaps by devising some judicious ansatz, at least some of the solutions can be obtained. It would be extremely interesting if in such an attempt we need to discover some “integrable structure” for the diagonalization of the spectrum, just as in the super-Yang-Mills case.

Some preliminary investigations in these directions are underway and we hope to report our progress elsewhere.

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Appendix A: On the orbital $su(4)$ generator l^i_j

In this appendix, we will elaborate on the orbital $su(4)$ generator l^i_j , which plays a crucial role in the analysis of the allowed representations of $psu(2,2|4)$ for our system.

According to the general formula (3.6), the full $su(4)$ Noether charge \mathbb{J}^i_j is given by

$$\mathbb{J}^i_j = \frac{1}{e}(G^{-1}J^i_j G)^a J^a_B. \quad (\text{A.1})$$

Consider the part of $G^{-1}J^i_j G$ independent of the fermionic coordinates. Since g_x and g_ϕ commute with J^i_j , it collapses to the purely $su(4)$ expression $g_y^{-1}J^i_j g_y$, which takes value in $su(4)$. The orbital generator l^i_j is then defined as

$$l^i_j \equiv \frac{1}{ie}(g_y^{-1}J^i_j g_y)^{A'} J^{A'}_B. \quad (\text{A.2})$$

It is not difficult to check that under the Dirac bracket il^i_j satisfy the canonical $su(4)$ commutation relations $\{il^i_j, il^k_n\}_D = \delta^k_j(il^i_n) - \delta^i_n(il^k_j)$. Upon quantization, l^i_j satisfies the same form of the algebra.

Let us give a more explicit form of l^i_j . To this end, define a 4×4 matrix U_t depending on a parameter t as

$$U_t \equiv \exp\left(t \frac{i}{2} y^{A'} \gamma^{A'}\right). \quad (\text{A.3})$$

Then one can show the following relation

$$g_{-ty} J^i_j g_{ty} = (U_{-t})^i_k J^k_l (U_t)^l_j, \quad (\text{A.4})$$

where $g_{ty} \equiv \exp(ty^i_j J^j_i)$. Since the equality is trivial at $t = 0$, this can be proved by demonstrating that both sides satisfy the same first order differential equation with respect to t . Now set $t = 1$ and substitute the expression of J^i_j in terms of the $SO(6)$ generators, namely

$$J^i_j = \frac{1}{4}(\gamma^{A'B'})^i_j J^{A'B'} + \frac{i}{2}(\gamma^{A'})^i_j J^{A'}, \quad (\text{A.5})$$

into the right hand side of (A.4) and focus on the coset part proportional to $J^{A'}$. Then we obtain

$$(g_y^{-1} J^i_j g_y)^{A'} = \frac{i}{2} (U^{-1} \gamma^{A'} U)^i_j, \quad (\text{A.6})$$

where $U \equiv U_{t=1}$. Applying this formula to the definition of l^i_j we get

$$l^i_j = \frac{1}{2e} (U^{-1} \gamma^{A'} U)^i_j J_B^{A'} = \frac{1}{2} (U^{-1} \gamma^{A'} U)^i_j (\partial J / \partial \dot{X})^{-1}_{A'}{}^{B'} P_{B'}. \quad (\text{A.7})$$

In the second equality we expressed $J_B^{A'}$ in terms of the phase space variables. This form of l^i_j was utilized in the calculation of the Noether charges. Further, the explicit form of U can be easily computed:

$$U = \cos \frac{|y|}{2} + i \frac{\gamma^{A'} y^{A'}}{|y|} \sin \frac{|y|}{2}, \quad |y| \equiv \sqrt{y^{A'} y^{A'}}. \quad (\text{A.8})$$

With the form of l^i_j given in (A.7), it is not difficult to prove the important quadratic identity (3.10). First consider the classical case. Using (A.7) we have

$$l^i_j l^j_k = \frac{1}{4e^2} (U^{-1} \gamma^{A'} \gamma^{B'} U)^i_k J_B^{A'} J_B^{B'} = \frac{1}{4e^2} \delta_k^i (J_B^{A'} J_B^{A'}). \quad (\text{A.9})$$

Taking the trace with respect to the indices i, k we obtain $\hat{l}^2 \equiv l^i_j l^j_i = \frac{1}{e^2} (J_B^{A'} J_B^{A'})$. Putting this back into (A.9) we get the formula

$$l^i_j l^j_k = \frac{1}{4} \delta_k^i \hat{l}^2. \quad (\text{A.10})$$

The quantum version of l^i_j is given by the second part of the formula (A.7) with $P_{B'}$ understood as the differential operator $-i\partial/\partial y^{B'}$ and symmetrized as $\frac{1}{4}(V_j^{iB'} P_{B'} + P_{B'} V_j^{iB'})$, where $V_j^{iB'} = (U^{-1} \gamma^{A'} U)^i_j (\partial J / \partial \dot{X})^{-1}_{A'}{}^{B'}$. This is needed to realize the hermiticity property $(l^i_j)^\dagger = l^j_i$. Then due to the re-ordering, quantum version of the formula (A.10) acquires an extra term linear in l^i_j and reads

$$l^i_j l^j_k = \frac{1}{4} \delta_k^i \hat{l}^2 + 2l^i_j. \quad (\text{A.11})$$

Appendix B: List of classical Noether charges

In this appendix, we give the list of the classical Noether charges. The following notations are used: $(S_\eta)^2 = S_{\eta^i} S_{\eta_i}$, $(S_\theta)^2 = S_{\theta^i} S_{\theta_i}$.

$$\mathbb{P}^+ = iP_-, \quad (\text{B.1})$$

$$\mathbb{P}^x = iP_{\bar{x}}, \quad (\text{B.2})$$

$$\mathbb{P}^{\bar{x}} = iP_x, \quad (\text{B.3})$$

$$\mathbb{P}^- = iP_+ = \frac{i}{4P_-} \left[(-4P_x P_{\bar{x}} - e^{-2\phi} P_\phi^2 - e^{-2\phi} (l^2 + (S_\eta^2)^2 + 4l^i{}_j S_{\eta^j} S_{\eta_i}) \right], \quad (\text{B.4})$$

$$\mathbb{Q}^{+i} = -\sqrt{P_-} S_{\theta^i}, \quad (\text{B.5})$$

$$\mathbb{Q}^+{}_i = \sqrt{P_-} S_{\theta_i}, \quad (\text{B.6})$$

$$\mathbb{Q}^{-i} = \frac{1}{2\sqrt{P_-}} \left[-2P_x S_{\theta^i} + iP_\phi e^{-\phi} S_{\eta^i} + e^{-\phi} (S_{\eta^i} (S_\eta)^2 + 2l^i{}_k S_{\eta^k}) \right], \quad (\text{B.7})$$

$$\mathbb{Q}^-{}_i = -\frac{1}{2\sqrt{P_-}} \left[-2P_{\bar{x}} S_{\theta_i} - iP_\phi e^{-\phi} S_{\eta_i} + e^{-\phi} (S_{\eta_i} (S_\eta)^2 + 2S_{\eta^k} l^k{}_i) \right], \quad (\text{B.8})$$

$$\mathbb{S}^{+i} = \sqrt{P_-} (e^\phi S_{\eta^i} + i\bar{x} S_{\theta^i}) + i\tau \mathbb{Q}^{-i}, \quad (\text{B.9})$$

$$\mathbb{S}^+{}_i = -\sqrt{P_-} (e^\phi S_{\eta_i} - ix S_{\theta_i}) - i\tau \mathbb{Q}^-{}_i, \quad (\text{B.10})$$

$$\mathbb{S}^{-i} = \frac{1}{2\sqrt{P_-}} \left[2P_{\bar{x}} e^\phi S_{\eta^i} + 2S_{\eta^i} S_{\theta^k} S_{\eta_k} + S_{\theta^i} ((S_\theta)^2 - iP_\phi) + 2l^i{}_k S_{\theta^k} \right] + ix^- \mathbb{Q}^{+i} + ix \mathbb{Q}^{-i}, \quad (\text{B.11})$$

$$\mathbb{S}^-{}_i = -\frac{1}{2\sqrt{P_-}} \left[2P_x e^\phi S_{\eta_i} + 2S_{\eta_i} S_{\eta^k} S_{\theta_k} + S_{\theta_i} ((S_\theta)^2 + iP_\phi) + 2S_{\theta^k} l^k{}_i \right] - ix^- \mathbb{Q}^+{}_i - i\bar{x} \mathbb{Q}^-{}_i, \quad (\text{B.12})$$

$$\mathbb{K}^+ = \frac{1}{i} (e^{2\phi} + x\bar{x}) P_- + \tau (iP_\phi + ixP_x + i\bar{x}P_{\bar{x}} + \tau \mathbb{P}^-), \quad (\text{B.13})$$

$$\mathbb{K}^x = -ie^{2\phi} P_{\bar{x}} + x(iP_\phi + ix^- P_- + ixP_x + \frac{1}{2}(-(S_\theta)^2 + (S_\eta)^2)) - ie^\phi S_{\theta^k} S_{\eta_k} - \tau \mathbb{J}^{-x}, \quad (\text{B.14})$$

$$\mathbb{K}^{\bar{x}} = -ie^{2\phi} P_x + \bar{x}(iP_\phi + ix^- P_- + i\bar{x}P_{\bar{x}} + \frac{1}{2}((S_\theta)^2 - (S_\eta)^2)) - ie^\phi S_{\eta^k} S_{\theta_k} - \tau \mathbb{J}^{-\bar{x}}, \quad (\text{B.15})$$

$$\begin{aligned} \mathbb{K}^- &= (x\bar{x} - e^{2\phi}) \mathbb{P}^- + x \mathbb{J}^{-\bar{x}} + \bar{x} \mathbb{J}^{-x} + ix^- P_\phi + i(x^-)^2 P_- \\ &\quad + \frac{i}{4P_-} \left[-(S_\eta^2)^2 + (S_\theta^2)^2 + 4S_{\theta^k} S_{\eta_k} S_{\eta^l} S_{\theta_l} \right. \\ &\quad \left. + 4e^\phi (P_x S_{\theta^k} S_{\eta_k} + P_{\bar{x}} S_{\eta^k} S_{\theta_k}) + 4l^k{}_l (S_{\theta^l} S_{\theta_k} - S_{\eta^l} S_{\eta_k}) \right], \end{aligned} \quad (\text{B.16})$$

$$\mathbb{D} = -iP_\phi - (ix^- P_- + ixP_x + i\bar{x}P_{\bar{x}}) - \tau \mathbb{P}^-, \quad (\text{B.17})$$

$$\mathbb{J}^{+-} = -ix^- P_- + \tau \mathbb{P}^-, \quad (\text{B.18})$$

$$\mathbb{J}^{x\bar{x}} = -i\bar{x}P_{\bar{x}} + ixP_x + \frac{1}{2}((S_\eta)^2 - (S_\theta)^2), \quad (\text{B.19})$$

$$\mathbb{J}^{+x} = -ixP_- + i\tau P_{\bar{x}}, \quad (\text{B.20})$$

$$\mathbb{J}^{+\bar{x}} = -i\bar{x}P_- + i\tau P_x, \quad (\text{B.21})$$

$$\mathbb{J}^{-x} = -x\mathbb{P}^- + ix^- P_{\bar{x}} + \frac{P_{\bar{x}}}{2P_-}((S_\eta)^2 + (S_\theta)^2) - \frac{1}{\sqrt{P_-}}S_{\theta^k}\mathbb{Q}_k^-, \quad (\text{B.22})$$

$$\mathbb{J}^{-\bar{x}} = -\bar{x}\mathbb{P}^- + ix^- P_x - \frac{P_x}{2P_-}((S_\eta)^2 + (S_\theta)^2) - \frac{1}{\sqrt{P_-}}\mathbb{Q}^{-k}S_{\theta_k}, \quad (\text{B.23})$$

$$\mathbb{J}^i_j = l^i_j - (S_{\eta^i}S_{\eta_j} - \frac{1}{4}(S_\eta)^2\delta^i_j) - (S_{\theta^i}S_{\theta_j} - \frac{1}{4}(S_\theta)^2\delta^i_j). \quad (\text{B.24})$$

Appendix C: Analysis of the allowed $su(4)$ representations

In this appendix, we provide some details of the analysis of the allowed $su(4)$ representations sketched in the main text.

Let us first describe the analysis of the relations (4.35) valid on superconformal primaries. Just as in the analysis of the relation (4.25) for the orbital part, $\mathcal{J}^2_1 \approx 0$, $\mathcal{J}^3_2 \approx 0$ and $\mathcal{J}^4_3 \approx 0$ give the following 3 equations:

$$(i) \quad (\lambda_1 + 2\lambda_2 + \lambda_3 + N - 2)E_1^-|\lambda_1, \lambda_2, \lambda_3\rangle = 0, \quad (\text{C.1})$$

$$(ii) \quad (\lambda_1 - \lambda_3 + 4 - N)E_2^-|\lambda_1, \lambda_2, \lambda_3\rangle = 0, \quad (\text{C.2})$$

$$(iii) \quad (\lambda_1 + 2\lambda_2 + \lambda_3 + 6 - N)E_3^-|\lambda_1, \lambda_2, \lambda_3\rangle = 0. \quad (\text{C.3})$$

Here and hereafter, $|\lambda_1, \lambda_2, \lambda_3\rangle$ refers to the $su(4)$ highest weight state in the total Hilbert space $\mathcal{H}_{tot} = \mathcal{H}_{spin} \otimes \mathcal{H}_{orb}$, consisting of the spin part and the orbital part. When we need to emphasize this feature, we will denote the state as $|\lambda_1, \lambda_2, \lambda_3\rangle_{tot}$.

Consider the equation (i). For $N \geq 3$ the coefficient is non-vanishing and we must have $E_1^-|\lambda_1, \lambda_2, \lambda_3\rangle = 0$ and hence $\lambda_1 = 0$. Similarly, from (iii) we find $\lambda_3 = 0$ for $N \leq 5$. Therefore for $3 \leq N \leq 5$ the relation (ii) reduces to $(4 - N)E_2^-|0, \lambda_2, 0\rangle = 0$. This tells us that λ_2 is arbitrary for $N = 4$, while for $N = 3, 5$ only the singlet state $|0, 0, 0\rangle$ is allowed.

Next consider the cases with $N \leq 2$. We already know that $\lambda_3 = 0$. Thus (ii) becomes $(\lambda_1 + 4 - N)E_2^-|\lambda_1, \lambda_2, 0\rangle = 0$. But since $4 - N > 0$, we must have $E_2^-|\lambda_1, \lambda_2, 0\rangle = 0$ and hence $\lambda_2 = 0$. Then (i) reduces to $(\lambda_1 + N - 2)E_1^-|\lambda_1, 0, 0\rangle = 0$. From this we easily find

that the possible values of λ_1 are $\lambda_1 = 0$ for $N = 2$, $\lambda_1 = 0, 1$ for $N = 1$, and $\lambda_1 = 0, 2$ for $N = 0$. Actually for $N = 0$ only the singlet $|0, 0, 0\rangle$ is allowed since without exciting any fermionic oscillator we cannot produce the state $|2, 0, 0\rangle$. Since the analysis for the cases with $N \geq 6$ is very similar, it will be omitted.

Combining these results, one finds that at this point the following highest weight states are allowed: $|0, 0, 0\rangle_{tot}$ for $N = 0, 1, 2, 3, 5, 6, 7, 8$, $|1, 0, 0\rangle_{tot}$ for $N = 1$, $|0, 0, 1\rangle_{tot}$ for $N = 7$ and $|0, \lambda, 0\rangle_{tot}$ with arbitrary non-negative integer λ for $N = 4$.

These states must be realized as the tensor products of the spin part and the orbital part. The spin part is generated by the fermionic oscillators S^i and \tilde{S}^i . From the form of the spin part of the $su(4)$ generators given in (3.26), one can easily find the Dynkin labels carried by S^i and \tilde{S}^i . They are $[1, 0, 0]$ for S^1, \tilde{S}^1 , $[-1, 1, 0]$ for S^2, \tilde{S}^2 , $[0, -1, 1]$ for S^3, \tilde{S}^3 and $[0, 0, -1]$ for S^4, \tilde{S}^4 . As for the orbital part, we already know that the allowed highest weight states are of the form $|0, l, 0\rangle_{orb}$, with an arbitrary non-negative integer l . With this information, one can easily analyze the possible value of l for realizing the states $|0, 0, 0\rangle_{tot}$, $|1, 0, 0\rangle_{tot}$, $|0, 0, 1\rangle_{tot}$ and $|0, \lambda, 0\rangle_{tot}$ for each relevant value of N .

For example, for $N = 0$, *i.e.* without exciting any fermionic oscillators, the orbital part must be $|0, 0, 0\rangle_{orb}$ in order to realize $|0, 0, 0\rangle_{tot}$. As another example, consider the realization of the state $|0, 0, 1\rangle_{tot}$ for $N = 1$. With one fermion excited the highest weight state of the spin part carries the Dynkin index $[1, 0, 0]$. Thus to produce $|1, 0, 0\rangle_{tot}$, the orbital part must again be $|0, 0, 0\rangle_{orb}$. A slightly non-trivial example occurs for $N = 2$. With two fermionic oscillators excited, the spin part can be $S^1 S^2 |0\rangle$, which is the highest weight state with the index $[0, 1, 0]$. Thus it can produce $|0, 0, 0\rangle_{tot}$ when tensored with $|0, 1, 0\rangle_{orb}$. So in this case $l = 1$ is allowed.

As a result of this type of analysis, one obtains a more refined information for the allowed highest weight states. Let us summarize the result by listing the value of N , the allowed total highest weight, and the value of l for its orbital part:

$$\begin{aligned}
N = 0 : & \quad |0, 0, 0\rangle_{tot}, \quad l = 0, & N = 8 : & \quad |0, 0, 0\rangle_{tot}, \quad l = 0, \\
N = 1 : & \quad |1, 0, 0\rangle_{tot}, \quad l = 0, & N = 7 : & \quad |0, 0, 1\rangle_{tot}, \quad l = 0, \\
N = 2 : & \quad |0, 0, 0\rangle_{tot}, \quad l = 1, & N = 6 : & \quad |0, 0, 0\rangle_{tot}, \quad l = 1, \\
N = 3 : & \quad \text{no solution}, & N = 5 : & \quad \text{no solution}, \\
N = 4 : & \quad |0, \lambda_2, 0\rangle_{tot}, \quad l - 2 \leq \lambda_2 \leq l + 2, \quad \text{any } l.
\end{aligned} \tag{C.4}$$

To further reduce the possibilities, one can impose the condition of unitarity for the representation of $psu(2, 2|4)$ built upon these $su(4)$ states. As mentioned in the main

text, one can obtain several different bounds depending on the choice of the pair of supercharges. A particularly useful combination is the bound

$$E \geq \lambda_1 + \lambda_2 + \lambda_3, \quad (\text{C.5})$$

which is powerful enough to eliminate many of the possible states.

The information given in the list (C.4) is sufficient to compute the energy of the superconformal primaries based on these $su(4)$ highest weight states. From (4.37) the energy can be expressed as

$$\mathbb{E} = \frac{1}{4}(\hat{\mathbb{J}}^2 - \hat{l}^2) + \frac{1}{16}(N - 4)^2 - 1. \quad (\text{C.6})$$

The general formula for the value of the Casimir operator $\hat{\mathbb{J}}^2$ on the highest weight state $|\lambda_1, \lambda_2, \lambda_3\rangle_{tot}$ reads¹³

$$\begin{aligned} & \hat{\mathbb{J}}^2 |\lambda_1, \lambda_2, \lambda_3\rangle_{tot} \\ &= \left(\frac{1}{4}(3\lambda_1^2 + 2\lambda_1\lambda_3 + 3\lambda_3^2) + (\lambda_2 + 3)(\lambda_1 + \lambda_3) + \lambda_2(\lambda_2 + 4) \right) |\lambda_1, \lambda_2, \lambda_3\rangle_{tot}. \end{aligned} \quad (\text{C.7})$$

The value of \hat{l}^2 on $|0, \lambda, 0\rangle_{orb}$ was already quoted in the main text to be $\lambda(\lambda + 4)$, which is actually a special case of (C.7). Using these formulas, we can readily compute the AdS energy of the superconformal primary state which can be built upon the states listed above. If we denote the energy for the state with N fermionic oscillators excited by E_N , the result is

$$E_0 = E_8 = 0, \quad E_1 = E_7 = \frac{1}{2}, \quad E_2 = E_6 = -2, \quad (\text{C.8})$$

$$E_4 = \frac{1}{4}(\lambda_2(\lambda_2 + 4) - l(l + 4)) - 1. \quad (\text{C.9})$$

On the other hand, the bounds following from (C.5) are, $E \geq 0$ for $N = 0, 2, 6, 8$, $E \geq 1$ for $N = 1, 7$, and $E \geq \lambda_2$ for $N = 4$. Evidently, the cases for $N = 1, 2, 6, 7$ are excluded, while the cases for $N = 0, 8$ are allowed. As for the case with $N = 4$, it is easy to see that the bound $E_4 \geq \lambda_2$ reduces to $\lambda_2 \geq l + 2$ and hence it is allowed for $\lambda_2 = l + 2$. One can check that these allowed cases actually meet all the other bounds as well.

¹³This formula can be easily derived by using the expression of \mathbb{J}^i_j in the Chevalley basis, just as in (4.29).

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